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Pinned Brownian Motion and its Perturbations

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1. PINNED BROWNIAN MOTION ON THE FRAME BUNDLE

Throughout, M will be a compact, connected, d-dimensional Riemannian manifold, and $\mathcal{O}(M)$[1] with fiber map $\pi: \mathcal{O}(M) \to M$ will denote the associated bundle of orthonormal frames $\mathfrak{e}: \mathbb{R}^d \to T_{\pi(\mathfrak{e})}(M)$. Further, we use the Lévi-Civita connection to determine the horizontal subspace $H_{\mathfrak{e}}(\mathcal{O}(M))$ in $T_{\mathfrak{e}}(\mathcal{O}(M))$ at each $\mathfrak{e} \in \mathcal{O}(M)$; and, for each $\mathbf{v} \in \mathbb{R}^d$, we define the basic vector field $\mathfrak{E}(\mathbf{v})$ on $\mathcal{O}(M)$ so that, for each $\mathfrak{e} \in \mathcal{O}(M)$,

$$\mathfrak{E}(\mathbf{v})_{\mathfrak{e}} \in H(\mathcal{O}(M)) \qquad \text{and} \qquad d\pi\,\mathfrak{E}(\mathbf{v})_{\mathfrak{e}} = \mathfrak{e}\mathbf{v}.$$

In particular, if $(\mathbf{e}_1, ..., \mathbf{e}_d)$ is the standard basis in $\mathbb{R}^d$, we set $\mathfrak{E}_k(\mathfrak{e}) = \mathfrak{E}(\mathbf{e}_k)_{\mathfrak{e}}$.

Next, given an element $\mathcal{O}$ of the orthogonal group $\mathcal{O}(d)$, we define the right action $\mathbf{R}_{\mathcal{O}}$ of $\mathcal{O}$ on $\mathcal{O}(M)$ so that $\pi \circ \mathbf{R}_{\mathcal{O}} = \pi$ and, for each $\mathfrak{e} \in \mathcal{O}(M)$, $\mathbf{R}_{\mathcal{O}}\mathfrak{e}$ is determined by $\mathbf{R}_{\mathcal{O}}\mathfrak{e}\mathbf{v} = \mathfrak{e}\mathcal{O}\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^d$. A Borel probability measure ν on $\mathcal{O}(M)$ is said to be *rotation invariant* if $\nu = \nu \circ \mathbf{R}_{\mathcal{O}}^{-1}$ for every $\mathcal{O} \in O(d)$. Equivalently, ν is rotation invariant if, for ν-almost every $x \in M$, the conditional distribution ν_x of ν given the base point x is *uniform* on the fiber $\pi^{-1}(x)$ in the sense that, for any $\mathfrak{e} \in \pi^{-1}(x)$, ν_x is the image under $\mathcal{O} \in O(d) \mapsto \mathbf{R}_{\mathcal{O}}\mathfrak{e} \in \pi^{-1}(x)$ of the normalized Haar measure $\lambda_{\mathcal{O}(d)}$ on $\mathcal{O}(d)$. In particular, if λ_M is the normalized Riemannian measure on M, then there

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[1] In [ES], we used $F(M)$ instead of $\mathcal{O}(M)$.

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is a unique, rotation invariant $\lambda_{O(d)}$ whose marginal distribution (i.e., image under π) on M is λ_M. Namely, given any measurable section $\mathfrak{e}\colon U \to \mathcal{O}(M)$,

$$\int_{\mathcal{O}(M)} \varphi \, d\lambda_{\mathcal{O}(M)} = \int_M \left(\int_{O(d)} \varphi(\mathbf{R}_{\mathcal{O}} \mathfrak{e}(x)) \, \lambda_{O(d)}(d\mathcal{O}) \right) \lambda_M(dx)$$

for measurable $\varphi\colon \mathcal{O}(M) \to [0, \infty)$.

1.1. Lemma. *For every* $\mathbf{v} \in \mathbb{R}^d$ *and* $\varphi_1, \varphi_2 \in C^1(\mathcal{O}(M))$,

$$\int_{\mathcal{O}(M)} \varphi_1 \mathfrak{E}(\mathbf{v}) \, \varphi_2 \, d\lambda_{\mathcal{O}(M)} = -\int_{\mathcal{O}(M)} \varphi_2 \mathfrak{E}(\mathbf{v}) \, \varphi_1 \, d\lambda_{\mathcal{O}(M)}. \tag{1.2}$$

Proof. There are lots of ways to prove (1.2), of which the one used here is the most pedestrian. Namely, we will show that, for each $\mathfrak{e} \in \mathcal{O}(M)$, there is a coordinate system at $\mathfrak{e}$ in which the formal adjoint of $\mathfrak{E}(\mathbf{v})$ is equal to $-\mathfrak{E}(\mathbf{v})$ at $\mathfrak{e}$. In particular, because the problem is completely local, we may and will assume that $\mathcal{O}(M) = M \times O(d)$ in such a way that $\mathfrak{e} = (x, \mathbf{I}) \in M \times O(d)$.

First choose an open coordinate chart $(U, \boldsymbol{\xi})$ in M so that $\boldsymbol{\xi}$ is normal at x. Next, choose an open coordinate chart $(V, \boldsymbol{\eta})$ at $\mathbf{I}$ in $O(d)$. Clearly, we can arrange the coordinate chart $(U \times V, \boldsymbol{\xi} \times \boldsymbol{\eta})$ at $(x, \mathbf{I})$ in $\mathcal{O}(M)$ so that $(x, \mathbf{I})$ corresponds to $(\mathbf{0}, \mathbf{0}) \in \mathbb{R}^d \times \mathbb{R}^{(1/2)d(d-1)}$, in which case

$$\mathfrak{E}(\mathbf{v})_{(\boldsymbol{\xi}, \boldsymbol{\eta})} = \sum_{k=1}^{d} \alpha_k(\boldsymbol{\xi}, \boldsymbol{\eta}) \frac{\partial}{\partial \xi_k} + \sum_{l=1}^{(1/2)d(d-1)} \beta_l(\boldsymbol{\xi}, \boldsymbol{\eta}) \frac{\partial}{\partial \boldsymbol{\eta}_l},$$

and

$$\lambda_{\mathcal{O}(M)}(d\boldsymbol{\xi} \times d\boldsymbol{\eta}) = f(\boldsymbol{\xi}, \boldsymbol{\eta}) \, d\boldsymbol{\xi} \times d\boldsymbol{\eta},$$

where

$$|\alpha_k(\boldsymbol{\xi}, \mathbf{0}) - v_k| \leqslant C \, |\boldsymbol{\xi}|^2, \qquad |\beta_l(\mathbf{0}, \boldsymbol{\eta})| \leqslant C \, |\boldsymbol{\eta}|^2,$$

and

$$|f(\boldsymbol{\xi}, \mathbf{0}) - f(\mathbf{0}, \mathbf{0})| \leqslant C \, |\boldsymbol{\xi}|^2$$

for some $C < \infty$ and all $(\boldsymbol{\xi}, \boldsymbol{\eta})$ sufficiently close to the origin. Hence, in this coordinate system, the action of the formal adjoint of $\mathfrak{E}(\mathbf{v})$ on a function φ at $(x, \mathbf{I})$ is given by

$$-\frac{1}{f(\mathbf{0}, \mathbf{0})} \left[\sum_{k=1}^{d} \frac{\partial (f \alpha_k \varphi)}{\partial \xi_k} - \sum_{l=1}^{(1/2)d(d-1)} \frac{\partial (f \beta_l \varphi)}{\partial \eta_l} \right] (\mathbf{0}, \mathbf{0}))$$
$$= -\sum_{k=1}^{d} v_k \frac{\partial \varphi}{\partial \xi_k} (\mathbf{0}, \mathbf{0}) = -\mathfrak{E}(\mathbf{v})_{(x, \mathbf{I})} \varphi. \quad \blacksquare$$

Define the gradient ∇ and the Laplace Δ operators by

$$\nabla\varphi = \sum_{k=1}^{d} (\mathfrak{E}_k \varphi)\mathbf{e}_k \qquad \text{and} \qquad \Delta\varphi = \sum_{k=1}^{d} \mathfrak{E}_k^2 \varphi \qquad \text{for} \quad \varphi \in C^2(\mathcal{O}(M); \mathbb{R})$$

$$\nabla\varphi = \nabla(\varphi \circ \pi) \qquad \text{and} \qquad \Delta\varphi = \Delta(\varphi \circ \pi) \qquad \text{for} \quad \varphi \in C^2(M; \mathbb{R}). \tag{1.3}$$

Because

$$\mathfrak{E}_k(\mathbf{R}_{\mathcal{O}}\,\mathfrak{e}) = \sum_{l=1}^{d} \mathcal{O}_{l,k}\, d\mathbf{R}_{\mathcal{O}}\mathfrak{E}_l(\mathfrak{e}), \tag{1.4}$$

it is easy to see that $\Delta\varphi$ is well-defined on M when $\varphi \in C^2(M; \mathbb{R})$. In particular, as a consequence of (1.2), we see that

$$\begin{gathered}\int_{\mathcal{O}(M)} \varphi_1\, \Delta\varphi_2\, d\lambda_{\mathcal{O}(M)} = \int_{\mathcal{O}(M)} \varphi_2\, \Delta\varphi_1\, d\lambda_{\mathcal{O}(M)} \qquad \text{if} \quad \varphi_1, \varphi_2 \in C^2(\mathcal{O}(M); \mathbb{R}) \\ \int_{M} \varphi_1\, \Delta\varphi_2\, d\lambda_{M} = \int_{M} \varphi_2\, \Delta\varphi_1\, d\lambda_{M} \qquad \text{if} \quad \varphi_1, \varphi_2 \in C^2(M; \mathbb{R}).\end{gathered} \tag{1.5}$$

Before describing the diffusion process corresponding to $\frac{1}{2}\Delta$, we have to introduce the path spaces $\mathscr{P}(\mathcal{O}(M)) \equiv C([0, \infty); \mathcal{O}(M))$ and $\mathscr{P}(M) \equiv C([0, \infty); M)$, and think of these as Polish spaces with the topology of uniform convergence on compacts. Further, for each $t \in [0, \infty)$, we use $\mathscr{B}_t(\mathcal{O}(M))$ and $\mathscr{B}_t(M)$ to denote the sigma-algebras over $\mathscr{P}(\mathcal{O}(M))$ and $\mathscr{P}(M)$ generated by the restriction of the path to $[0, t]$; and when it is clear from the context which one we mean, we use $\mathscr{B}_t$ to stand for either $\mathscr{B}_t(\mathcal{O}(M))$ or $\mathscr{B}_t(M)$. Finally, $\mathbf{M}_1(\mathscr{P}(\mathcal{O}(M)))$ and $\mathbf{M}_1(\mathscr{P}(M))$ will denote the spaces of Borel probability measures on $\mathscr{P}(\mathcal{O}(M))$ and $\mathscr{P}(M)$, respectively; and we will think of these as Polish spaces with the topology of weak convergence.

1.6. THEOREM. *For each $\mathfrak{e} \in \mathcal{O}(M)$ there is a unique $\mathfrak{P}_{\mathfrak{e}} \in \mathbf{M}_1(\mathscr{P}(\mathcal{O}(M)))$ with the properties that* $\mathfrak{P}_{\mathfrak{e}}(\mathfrak{p}(0) = \mathfrak{e}) = 1$ and

$$\left(\varphi(\mathfrak{p}(t)) - \int_0^t \tfrac{1}{2}[\Delta\varphi](\mathfrak{p}(\tau))\, d\tau, \mathscr{B}_t(\mathcal{O}(M)), \mathfrak{P}_{\mathfrak{e}}\right)$$

$$\textit{is a martingale for every } \varphi \in C^2(\mathcal{O}(M); \mathbb{R}). \tag{1.7}$$

Moreover, for each $\mathcal{O} \in \mathcal{O}(d)$, $\mathfrak{P}_{\mathbf{R}_{\mathcal{O}}\mathfrak{e}}$ is the distribution of $\mathfrak{p} \rightsquigarrow \mathbf{R}_{\mathcal{O}}\mathfrak{p}$ under $\mathfrak{P}_{\mathfrak{e}}$. In particular, for each $x \in M$, the distribution of $\mathfrak{p} \in \mathscr{P}((\mathcal{O}(M))) \mapsto \pi \circ \mathfrak{p} \in \mathscr{P}(M)$ under $\mathfrak{P}_{\mathfrak{e}}$ is independent of the choice of $\mathfrak{e} \in \pi^{-1}(x)$ and is, in *fact, the unique* $P_x \in \mathbf{M}_1(\mathscr{P}(M))$ *with the properties that* $P_x(p(0)=x)=1$ *and*

$$\left(\varphi(p(t)) - \int_0^t \tfrac{1}{2}[\Delta\varphi](p(\tau))\,d\tau, \mathscr{B}_t(M), P_x\right)$$

$$\text{is a martingale for every } \varphi \in C^2(M;\mathbb{R}). \tag{1.8}$$

Finally, $\mathfrak{e} \rightsquigarrow \mathfrak{P}_\mathfrak{e}$ *and* $x \rightsquigarrow P_x$ *are both continuous maps, and, depending on whether* ν *is a probability measure on* $\mathcal{O}(M)$ *or* M*, we set*

$$\mathfrak{P}_\nu \equiv \int_{\mathcal{O}(M)} \mathfrak{P}_\mathfrak{e}\,\nu(d\mathfrak{e}) \qquad \text{or} \qquad P_\nu \equiv \int_M P_x\,\nu(dx).$$

Then $\mathfrak{P}_{\mathcal{O}(M)} \equiv \mathfrak{P}_{\lambda_{\mathcal{O}(M)}}$ *and* $P_M \equiv P_{\lambda_M}$ *are reversible in the sense that the distribution of* $\{\mathfrak{p}(t): t\in[0,T]\}$ *and* $\{\mathfrak{p}(T-t): t \in [0,T]\}$ *have the same distribution under* $\mathfrak{P}_{\mathcal{O}(M)}$ *and similarly for* $p(\cdot)$ *under* P_M.

Proof. In spite of the length of the statement here, there is hardly anything to prove. Indeed, all the existence and uniqueness assertions are completely standard consequences of the martingale characterization of diffusions (cf. [SV]); and the continuity statements follow easily from uniqueness. Furthermore, the identification as $P_{\pi(\mathfrak{e})}$ of the distribution under $\mathfrak{P}_\mathfrak{e}$ of $\mathfrak{p} \rightsquigarrow \pi\circ\mathfrak{p}$ comes from uniqueness together with the fact that, for $\varphi \in C^2(M;\mathbb{R})$, $\Delta\varphi$ is well-defined on M. Thus, reversibility is the only item remaining unchecked. However, starting from (1.6), one can use standard semigroup considerations to check that the operators in the Markov semigroup generated by $\frac{1}{2}\Delta$ are self-adjoint in $L^2(\lambda_{\mathcal{O}(M)};\mathbb{R})$. Equivalently,

$$\mathbb{E}^{\mathfrak{P}_{\mathcal{O}(M)}}[\varphi_1(\mathfrak{p}(0))\,\varphi_2(\mathfrak{p}(t))] = \mathbb{E}^{\mathfrak{P}_{\mathcal{O}(M)}}[\varphi_2(\mathfrak{p}(0))\,\varphi_1(\mathfrak{p}(t))]$$

for all $t\in(0,\infty)$ and $\varphi_1,\varphi_2 \in C(\mathcal{O}(M);\mathbb{R})$. Next, using the Markov property and working by induction, one shows that, for all $n \geqslant 1$, $0=t_0<t_1<\cdots<t_n=T$, and $\Psi \in C(\mathcal{O}(M)^{n+1};\mathbb{R})$:

$$\begin{aligned}&\mathbb{E}^{\mathfrak{P}_{\mathcal{O}(M)}}[\Psi(\mathfrak{p}(t_0),\mathfrak{p}(t_1),\ldots,\mathfrak{p}(t_{n-1}),\mathfrak{p}(t_n))]\\ &\quad= \mathbb{E}^{\mathfrak{P}_{\mathcal{O}(M)}}[\Psi(\mathfrak{p}(T-t_0),\mathfrak{p}(T-t_1),\ldots,\mathfrak{p}(T-t_{n-1}),\mathfrak{p}(T-t_n))],\end{aligned}$$

which is equivalent to the stated reversibility. ∎

By standard elliptic regularity results (cf. Chapter 5 of [D]), one knows that there is a smooth map $(t,x,y)\in(0,\infty)\times M\times M \mapsto p_t(x,y)\in(0,\infty)$ with the property that

$$\mathbb{E}^{P_x}[\varphi(p(t))] = \int_M \varphi(y)\,p_t(x,y)\,\lambda_M(dy),$$

$$(t,x)\in(0,\infty)\times M \qquad \text{and} \qquad \varphi\in C(M;\mathbb{R}). \tag{1.9}$$

Moreover, from the reversibility statement for P_M, it is an easy step to the conclusion that $p_t(x, y) = p_t(y, x)$. Now, given an $\varepsilon \in (0, 1]$, choose $\eta_\varepsilon \in C^\infty(M^2; [0, \infty))$ so that

$$\eta_\varepsilon(x, x) > 0 \qquad \text{and} \qquad \eta_\varepsilon(x, y) = 0 \qquad \text{whenever} \quad \operatorname{dist}(x, y) \geqslant \varepsilon,$$

determine $K_\varepsilon(x, y) \in (0, \infty)$ by

$$\frac{1}{K_\varepsilon(x, y)} = \iint_{M^2} \eta_\varepsilon(x, z)\, p_1(z, z')\, \eta_\varepsilon(y, z')\, \lambda_M(dz)\, \lambda_M(dz'),$$

and define

$$\mathfrak{P}^\varepsilon_{(x, y)}(A) = \mathbb{E}^{\mathfrak{P}_M}[\rho_\varepsilon((x, \mathfrak{p}(0)), (y, \mathfrak{p}(1))), A],$$
$$(x, y) \in M^2 \qquad \text{and} \qquad A \in \mathscr{B}_1(\mathscr{O}(M)), \tag{1.10}$$

where

$$\rho_\varepsilon((x, \mathfrak{e}), (y, \mathfrak{e}')) \equiv K_\varepsilon(x, y)\, \eta_\varepsilon(x, \pi(\mathfrak{e}))\, \eta_\varepsilon(y, \pi(\mathfrak{e}')).$$

Finally, use $\mathscr{P}_1(\mathscr{O}(M))$ to denote the Polish space of paths $\mathfrak{p} \upharpoonright [0, 1]$ as $\mathfrak{p}$ varies over $\mathscr{P}(\mathscr{O}(M))$, take $\mathbf{M}_1(\mathscr{P}_1(\mathscr{O}(M)))$ to be the Polish space of probability measures on $\mathscr{P}_1(\mathscr{O}(M))$, and think of $\mathfrak{P}^\varepsilon_{(x, y)}$ as an element of $\mathbf{M}_1(\mathscr{P}(\mathscr{O}(M)))$.

1.11. Lemma. *The family*

$$\{\mathfrak{P}^\varepsilon_{(x, y)} : \varepsilon \in (0, 1] \text{ and } (x, y) \in M^2\}$$

is tight in $\mathbf{M}_1(\mathscr{P}_1(\mathscr{O}(M)))$. *Moreover, if* $\mathfrak{p} \in \mathscr{P}_1(\mathscr{O}(M)) \mapsto \check{\mathfrak{p}} \in \mathscr{P}_1(\mathscr{O}(M))$ *is defined by*

$$\check{\mathfrak{p}}(t) = \mathfrak{p}(1 - t), \quad t \in [0, 1], \quad \text{and} \quad \check{A} = \{\mathfrak{p} \in \mathscr{P}_1(\mathscr{O}(M)) : \check{\mathfrak{p}} \in A\}, \quad A \in \mathscr{B}_1, \tag{1.12}$$

then

$$\mathfrak{P}^\varepsilon_{(x, y)}(\check{A}) = \mathfrak{P}^\varepsilon_{(y, x)}(A), \qquad (x, y) \in M^2 \qquad \text{and} \qquad A \in \mathscr{B}_1. \tag{1.13}$$

Proof. The equality in (1.13) is an immediate consequence of the reversibility of $\mathfrak{P}_M$. Moreover, given (1.13), the proof of tightness comes down to showing that, for each $\alpha \in (0, 1)$, there is a compact subset K_α of $C([0, \frac{3}{4}]; \mathscr{O}(M))$ with the property that

$$\mathfrak{P}^\varepsilon_{(x, y)}(\{\mathfrak{p} : \mathfrak{p} \upharpoonright [0, \tfrac{3}{4}] \in K_\alpha\}) \geqslant 1 - \alpha \qquad \text{for} \quad \varepsilon \in (0, 1] \quad \text{and} \quad (x, y) \in M^2. \tag{1.14}$$

Indeed, if

$$A_\alpha \equiv \{\mathfrak{p} : \mathfrak{p} \restriction [0, \tfrac{3}{4}] \in K_\alpha \text{ and } \check{\mathfrak{p}} \restriction [0, \tfrac{3}{4}] \in K_\alpha\},$$

then A_α is a compact subset of $\mathscr{P}_1(\mathcal{O}(M))$ and (1.13) and (1.14) combine to yield

$$\mathfrak{P}^\varepsilon_{(x,y)}(A_\alpha) \geqslant 1 - 2\alpha \qquad \text{for all} \quad \varepsilon \in (0, 1] \qquad \text{and} \qquad (x, y) \in M^2.$$

To find a K_α for which (1.14) holds, we need to observe that, by the Markov property, for each $T \in [0, 1)$ and $A \in \mathscr{B}_T$:

$$\mathfrak{P}^\varepsilon_{(x,y)}(A) = \iint_{\mathcal{O}(M)^2} \rho_\varepsilon((x, \mathfrak{e}), (y, \mathfrak{e}'))\, \mathbb{E}^{\mathfrak{P}_\mathfrak{e}}[p_{1-T}(\pi \circ \mathfrak{p}(T), \pi(\mathfrak{e}')), A]$$

$$\times \lambda_{\mathcal{O}(M)}(d\mathfrak{e})\, \lambda_{\mathcal{O}(M)}(d\mathfrak{e}'). \tag{1.15}$$

In particular, because p_t achieves its maximum on the diagonal, if

$$C \equiv \sup_{(x, y) \in M2} \frac{p_{1/4}(y, y)}{p_1(x, y)},$$

then

$$\mathfrak{P}^\varepsilon_{(x,y)}(A) \leqslant C \sup_{\mathfrak{e} \in \mathcal{O}(M)} \mathfrak{P}_\mathfrak{e}(A), \qquad A \in \mathscr{B}_{3/4}.$$

Hence, since $\{\mathfrak{P}_\mathfrak{e} : \mathfrak{e} \in \mathcal{O}(M)\}$ is tight, we are done. ∎

1.16. THEOREM. *Given $x \in M$ and $\mathfrak{e} \in \pi^{-1}(x)$, set*

$$\mathfrak{P}_x = \int_{\mathcal{O}(d)} (\mathfrak{P}_{\mathbf{R}_\mathcal{O}\mathfrak{e}} \restriction \mathscr{B}_1)\, \lambda_{\mathcal{O}(d)}(d\mathcal{O}). \tag{1.17}$$

Then $\mathfrak{P}_x$ does not depend on the choice of $\mathfrak{e} \in \pi^{-1}(x)$. Moreover, there is a continuous map $(x, y) \in M^2 \mapsto \mathfrak{P}_{(x,y)} \in \mathbf{M}_1(\mathscr{P}_1(\mathcal{O}(M)))$ with the properties that

$$\mathfrak{P}_{(x,y)}(\pi \circ \mathfrak{p}(0) = x \textit{ and } \pi \circ \mathfrak{p}(1) = y) = 1 \tag{1.18}$$

and

$$\mathfrak{P}_x = \int_M p_1(x, y)\, \mathfrak{P}_{(x,y)}\, \lambda_M(dy). \tag{1.19}$$

In fact (cf. (1.12))

$$\mathfrak{P}_{(x,y)}(\check{A}) = \mathfrak{P}_{(y,x)}(A), \qquad A \in \mathscr{B}_1, \tag{1.20}$$

and, for each $T \in [0, 1)$:

$$\mathfrak{P}_{(x,y)}(A) = \mathbb{E}^{\mathfrak{P}_x}\left[\frac{p_{1-T}(\pi \circ \mathfrak{p}(T), y)}{p_1(x, y)}, A\right], \qquad A \in \mathscr{B}_T. \tag{1.21}$$

Finally, for each $T \in [0, 1]$ *and measurable section* $\mathfrak{e}: M \to \mathcal{O}(M)$,

$$\mathbb{E}^{\mathfrak{P}_{(x,y)}}[\varphi(\mathfrak{p}(T))] = \int_{O(d)} \mathbb{E}^{\mathfrak{P}_{(x,y)}}[\varphi(R_{\mathcal{O}}\,\mathfrak{e}(\pi \circ \mathfrak{p}(T)))]\, \lambda_{O(d)}(d\mathcal{O}), \quad \varphi \in C(\mathcal{O}(M)). \tag{1.22}$$

Proof. To prove the existence of continuity of $(x, y) \rightsquigarrow \mathfrak{P}_{(x,y)}$, let $\{(x_n, y_n)\}_1^\infty \subseteq M^2$ and $\{\varepsilon_n\}_1^\infty \subseteq (0, 1]$ satisfying $(x_n, y_n) \to (x, y)$ and $\varepsilon_n \searrow 0$ be given, and set $\mathfrak{P}_n = P^{\varepsilon_n}_{(x_n, y_n)}$. By Lemma 1.11, $\{\mathfrak{P}_n\}_1^\infty$ is relatively compact in $\mathbf{M}_1(\mathscr{P}_1(\mathcal{O}(M)))$. Moreover, from (1.15), we see that any limit of this sequence will satisfy (1.21) for all $T \in [0, 1)$. Hence, the sequence converges to a limit $\mathfrak{P}_{(x,y)}$ which is uniquely determined by the equality in (1.21). In particular, we conclude from this that $(x, y) \rightsquigarrow \mathfrak{P}_{(x,y)}$ is continuous. In addition, from (1.13), we see that (1.20) holds; and, from (1.15), it is clear that $\mathfrak{B}_{(x,y)}(\mathfrak{p}(0) = x) = 1$, which, together with (1.20), means that (1.18) holds. To prove (1.19), note that it suffices to check the equality for $A \in \bigcup_{T \in [0,1)} \mathscr{B}_T$, and observe that, when $A \in \mathscr{B}_T$, (1.21) implies (1.19). Finally, to prove (1.22), notice that the reversibility of $\mathfrak{P}_M$ guarantees that $\lambda_{\mathcal{O}(M)}$ is the distribution of $\mathfrak{p} \rightsquigarrow \mathfrak{p}(T)$ under $\mathfrak{P}_M$ for every T; and this leads to (1.22) first when $\mathfrak{P}_{(x,y)}$ is replaced by $\mathfrak{P}^{\varepsilon}_{(x,y)}$ and then, after $\varepsilon \searrow 0$, to (1.22) itself. ∎

Note that (1.18) and (1.19) combine to say that $y \in M \mapsto \mathfrak{P}_{(x,y)} \in \mathbf{M}_1(\mathscr{P}_1(\mathcal{O}(M)))$ is a continuous, regular conditional distribution of $\mathfrak{P}_x$ given that $\mathfrak{p}(1) = y$, and that (1.22) says that, for each $T \in [0, 1]$, the conditional distribution under $\mathfrak{P}_{(x,y)}$ of $\mathfrak{p}(T)$ given that $\pi \circ \mathfrak{p}(T)$ is uniform on the fiber. For obvious reasons, we call the measure $\mathfrak{P}_{(x,y)}$ the distribution of *pinned Brownian paths on* $\mathcal{O}(M)$.

In preparation for the concluding result in this section, we give the following characterization of $\mathfrak{P}_{(x,y)}$.

1.23. Corollary. *Set*

$$\mathfrak{K}_{t,y}(\mathfrak{e}) = \sum_{k=1}^{d} \frac{\mathfrak{E}_k(\mathfrak{e})\, p_{1-t}(\cdot, y)}{p_{1-t}(\pi(\mathfrak{e}), y)}\, \mathfrak{E}_k(\mathfrak{e}), \qquad (t, \mathfrak{e}, y) \in [0, 1) \times \mathcal{O}(M) \times M, \tag{1.24}$$

and define the operator $\mathscr{L}_{t,y}$ *for* $(t, y) \in [0, 1) \times M$ *so that*

$$\mathscr{L}_{t,y}\varphi = \tfrac{1}{2}\Delta\varphi + \mathfrak{K}_{t,y}\varphi, \qquad \varphi \in C^2(\mathcal{O}(M); \mathbb{R}). \tag{1.25}$$

Then, for each $(x, y) \in M^2$, $\mathfrak{P}_{(x,y)}$ *is the unique element of* $\mathbf{M}_1(\mathscr{P}_1(\mathcal{O}(M)))$ *with the properties that, for every* $\mathfrak{e} \in \pi^{-1}(x)$:

$$\mathfrak{P}_{(x,y)}(\mathfrak{p}(0) \in \Gamma) = \int_{\mathcal{O}(d)} \mathbf{1}_\Gamma(R_{\mathcal{O}}\mathfrak{e})\, \lambda_{\mathcal{O}(d)}(d\mathcal{O}), \qquad \Gamma \in \mathscr{B}_{\mathcal{O}(M)}, \tag{1.26}$$

and, for every $\varphi \in C^2(\mathcal{O}(M); \mathbb{R})$,

$$\left(\varphi(\mathfrak{p}(t)) - \int_0^t [\mathscr{L}_{\tau,y}\varphi](\mathfrak{p}(\tau))\, d\tau, \mathscr{B}_t, \mathfrak{P}_{(x,y)}\right), \qquad t \in [0, 1), \textit{ is a martingale.} \tag{1.27}$$

Proof. Since it is an easy matter to check that there is at most one element of $\mathbf{M}_1(\mathscr{P}_1(\mathcal{O}(M)))$ satisfying (1.26) and (1.27), all that remains is to show that $\mathfrak{P}_{(x,y)}$ is such an element. Moreover, (1.26) is an immediate consequence of (1.18) combined with (1.22). Finally, let φ be given, and set

$$f(t, \mathfrak{e}) = \varphi(\mathfrak{e})\, p_{1-t}(\pi(\mathfrak{e}), y) \qquad \text{for} \quad (t, \mathfrak{e}) \in [0, 1) \times \mathcal{O}(M).$$

Then

$$\left(f(t, \mathfrak{p}(t)) - \int_0^t \left[\frac{\partial f}{\partial \tau} + \frac{1}{2}\Delta f\right](\tau, \mathfrak{p}(\tau))\, d\tau, \mathscr{B}_t, \mathfrak{P}_x\right), \qquad t \in [0, 1)$$

is a martingale. Hence, since

$$\left[\frac{\partial f}{\partial \tau} + \frac{1}{2}\Delta f\right](\tau, \mathfrak{e}) = p_{1-\tau}(\pi(\mathfrak{e}), y)[\mathscr{L}_{\tau,y}\varphi](\tau, \mathfrak{e}),$$

the required result follows easily from (1.21). ∎

In order to state our next result, we need to introduce a little notation. Namely, let $(\mathbf{H}, \mathfrak{W}, \mu)$ be the standard Wiener space of $\mathbb{R}^d$-valued paths on $[0, 1]$. That is, $\mathfrak{W}$ is the Banach space of continuous $\mathbf{w}\colon [0, 1] \to \mathbb{R}^d$ with $\mathbf{w}(0) = \mathbf{0}$ and $\|\mathbf{w}\|_{\mathfrak{W}} = \sup_{t \in [0,1]} |\mathbf{w}(t)|$, $\mathbf{H}$ is the Hilbert space of $\mathbf{h} \in \mathfrak{W}$ with one derivative $\dot{\mathbf{h}} \in L^2([0, 1]; \mathbb{R}^d)$ and $\|\mathbf{h}\|_{\mathbf{H}} = \|\dot{\mathbf{h}}\|_{L2([0,1];\mathbb{R}^d)}$, and μ is the unique Borel measure on $\mathfrak{W}$ with the property that, for each α from the dual space $\mathfrak{W}^*$,

$$\begin{aligned}\hat{\mu}(\alpha) &\equiv \mathbb{E}^\mu[\exp(\sqrt{-1}\,\langle \mathbf{w}, \alpha\rangle)]\\ &= \exp\left[-\tfrac{1}{2}\iint_{[0,1]^2} \tau \wedge t(\alpha(d\tau), \alpha(dt))_{\mathbb{R}^d}\right],\end{aligned} \tag{1.28}$$

and we have identified $\mathfrak{W}^*$ with the space of totally finite, $\mathbb{R}^d$-valued Borel measures on $[0, 1]$. Next, set $\tilde{\mathfrak{W}} = \mathcal{O}(d) \times \mathfrak{W}$ and $\tilde{\mu} = \lambda_{\mathcal{O}(d)} \times \mu$, and, for $t \in [0, 1]$, take $\mathscr{B}_t$ to be the sigma-algebra over $\tilde{\mathfrak{W}}$ generated by the maps $\tilde{\mathbf{w}} = (\mathcal{O}, \mathbf{w}) \rightsquigarrow \mathcal{O}$ and $\tilde{\mathbf{w}} = (\mathcal{O}, \mathbf{w}) \rightsquigarrow \mathbf{w}(\tau)$ as τ runs over $[0, t]$.

The following lemma is an essentially trivial consequence of Corollary 1.23 and Itô's formula for the Stratonovich stochastic calculus.

1.29. COROLLARY. *Given* $(x, y) \in M^2$ *and* $\mathfrak{e} \in \pi^{-1}(x)$, *define* $\mathfrak{F}_{(\mathfrak{e}, y)}: [0, 1) \times \tilde{\mathfrak{W}} \to \mathcal{O}(M)$ *to be the* $\tilde{\mu}$*-almost surely unique* $\{\mathcal{B}_t : t \in [0, 1)\}$*-progressively measurable solution to the Stratonovich stochastic differential equation*

$$\mathfrak{F}_{(\mathfrak{e}, y)}(t, \tilde{\mathbf{w}}) = \sum_{k=1}^{d} \mathfrak{E}_k(\mathfrak{F}_{(\mathfrak{e}, y)}(t, \tilde{\mathbf{w}})) \circ d\mathbf{w}(t)_k + \mathfrak{K}_{t, y}(\mathfrak{F}_{(x, y)}(t, \tilde{\mathbf{w}}))\, dt,$$

$$t \in [0, 1), \qquad \text{and} \qquad \mathfrak{F}_{(x, y)}(0, \tilde{\mathbf{w}}) = \mathcal{O}\mathfrak{e}, \tag{1.30}$$

for $\tilde{\mu}$*-almost every* $\tilde{\mathbf{w}} = (\mathcal{O}, \mathbf{w})$. *Then,* $\tilde{\mu}$*-almost surely,*

$$\mathfrak{F}_{(x, y)}(1, \tilde{\mathbf{w}}) \equiv \lim_{t \nearrow 1} \mathfrak{F}_{(x, y)}(t, \tilde{\mathbf{w}}) \text{ exists,} \tag{1.31}$$

and $\mathfrak{P}_{(x, y)}$ *is the distribution of* $\tilde{\mathbf{w}} \in \tilde{\mathfrak{W}} \mapsto \mathfrak{F}_{(x, y)}(\cdot, \tilde{\mathbf{w}}) \in \mathscr{P}_1(\mathcal{O}(M))$ *under* $\tilde{\mu}$. *In particular,* $\pi \circ \mathfrak{F}_{(x, y)}(1, \tilde{\mathbf{w}}) = y$ for $\tilde{\mu}$*-almost every* $\tilde{\mathbf{w}}$.

2. THE PERTURBATION PROCEDURE

In this section we describe the ways in the perturbation procedure introduced in §2 of [ES] must be changed in order to carry out the same program for pinned Brownian paths. But before we can do so, we must recall some of the notation and ideas employed there.

The *solder form* $\omega: T(\mathcal{O}(M)) \to \mathbb{R}^d$ is the 1-form defined so that, for each $\mathfrak{e} \in \mathcal{O}(M)$ and $X_\mathfrak{e} \in T_\mathfrak{e}(\mathcal{O}(M))$, $d\pi\, X_\mathfrak{e} = \mathfrak{e}\omega(X_\mathfrak{e})$. Thus, the vertical subspace at $\mathfrak{e}$ is precisely the null space of $\omega \restriction T_\mathfrak{e}(\mathcal{O}(M))$. Next, let $\mathfrak{o}(d)$ stand for the Lie algebra of skew symmetric $d \times d$-matrices, remember that $\mathfrak{o}(d)$ can be identified with the Lie algebra of left-invariant vector fields on $O(d)$, and let λ be the map of $\mathfrak{o}(d)$ into the $T(\mathcal{O}(M))$ given by

$$\lambda(A)_\mathfrak{e} = \frac{d}{dt} R_{e^{tA}}\mathfrak{e}\Big|_{t=0}, \qquad A \in \mathfrak{o}(d) \qquad \text{and} \qquad \mathfrak{e} \in \mathcal{O}(M). \tag{2.1}$$

Clearly, $A \in \mathfrak{o}(d) \mapsto \lambda(A)_\mathfrak{e} \in T_\mathfrak{e}(\mathcal{O}(M))$ provides an isomorphism between $\mathfrak{o}(d)$ and the vertical subspace at $\mathfrak{e}$. Thus, we can define the *connection 1-form* $\phi: T(\mathcal{O}(M)) \to \mathfrak{o}(d)$ so that, for each $\mathfrak{e} \in \mathcal{O}(M)$ and $X_\mathfrak{e}(\mathcal{O}(M))$,

$$X_\mathfrak{e} - \lambda(\phi(X_\mathfrak{e})) = \sum_{k=1}^{d} \omega(X_\mathfrak{e})_k\, \mathfrak{E}_k(\mathfrak{e}) \text{ is the horizontal part of } X_\mathfrak{e}. \tag{2.2}$$

Equivalently, $\lambda(\phi(X_\mathfrak{e}))$ is the vertical part of $X_\mathfrak{e}$.

Next, for each $m \geqslant 1$, let $W_2^{(m)}(\mathbb{R}; \mathcal{O}(M))$ be the Sobolev space of $\mathfrak{f} \in C(\mathbb{R}; \mathcal{O}(M))$ with m square integrable derivatives. That is, $\mathfrak{f} \in C(\mathbb{R}; \mathcal{O}(M))$ is an element of $W_2^{(m)}(\mathbb{R}; \mathcal{O}(M))$ if $s \in \mathbb{R} \mapsto \omega(\mathfrak{f}'(s)) \in \mathbb{R}^d$ and $s \in \mathbb{R} \mapsto \phi(\mathfrak{f}'(s)) \in \mathfrak{o}(d)$ exist as elements of the classical Sobolev spaces $W_2^{(m-1)}(\mathbb{R}; \mathbb{R}^d)$ and $W_2^{(m-1)}(\mathbb{R}; \mathfrak{o}(d))$, respectively. It is obvious that $W_2^{(m)}(\mathbb{R}; \mathcal{O}(M))$ becomes a Polish space when we use the metric

$$\rho^{(m)}(\mathfrak{f}, \mathfrak{g}) = \operatorname{dist}(\mathfrak{f}(0), \mathfrak{g}(0)) + \|\omega(\mathfrak{f}') - \omega(\mathfrak{g}')\|_{W_2^{(m-1)}(\mathbb{R}; \mathbb{R}^d)} + \|\phi(\mathfrak{f}') - \phi(\mathfrak{g}')\|_{W_2^{(m-1)}(\mathbb{R}; \mathfrak{o}(d))}.$$

Moreover, we can determine a continuous map $\mathfrak{f} \in W_2^{(1)}(\mathbb{R}; \mathcal{O}(M)) \mapsto \mathcal{O}(\mathfrak{f}) \in C(\mathbb{R}; \mathcal{O}(d))$ by the integral equation

$$[\mathcal{O}(\mathfrak{f})](s) = \mathbf{I} - \int_0^s \phi(\mathfrak{f}'(\sigma))[\mathcal{O}(\mathfrak{f})](\sigma)\, d\sigma, \qquad \sigma \in \mathbb{R}, \tag{2.3}$$

and can introduce the associated map $\mathfrak{f} \in W_2^{(1)}(\mathbb{R}; \mathcal{O}(M)) \mapsto \mathscr{A}(\mathfrak{f}) \in W^{(1)}(\mathbb{R}; \mathbb{R}^d \otimes \mathbb{R}^d)$ given by

$$[\mathscr{A}(\mathfrak{f})](s) = [\mathcal{O}(\mathfrak{f})](s) \int_0^s [\mathcal{O}(\mathfrak{f})](\sigma)\, d\sigma. \tag{2.4}$$

The following lemma is simply a re-statement of Theorem 2.5 in [ES].

2.5. THEOREM. *Suppose that* $\mathbf{w}: [0, 1) \to \mathbb{R}^d$ *and* $\mathbf{h}: [0, 1) \to \mathbb{R}^d$ *are piece-wise smooth maps with* $\mathbf{w}(0) = \mathbf{0} = \mathbf{h}(0)$, *let* $\mathfrak{f} \in W_2^{(m)}(\mathbb{R}; \mathcal{O}(M))$, $\mathcal{O} \in \mathcal{O}(d)$, *and* $y \in M$ *be given, and set* $\tilde{\mathbf{w}} = (\mathcal{O}, \mathbf{w})$. *Then there exists a unique continuous map* $t \in [0, 1) \mapsto \mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}}) \in W_2^{(m)}(\mathbb{R}; \mathcal{O}(M))$ *such that* $[\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(\cdot, \tilde{\mathbf{w}})](s)$ *is piece-wise smooth for each* $s \in \mathbb{R}$ *and*

(a) $[\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(0, \tilde{\mathbf{w}})](s) = \mathbf{R}_{\mathcal{O}}\, \mathfrak{f}(s)$,

(b) $[\dot{\mathfrak{F}}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})](s)$ *is horizontal*,

(c) $\omega([\dot{\mathfrak{F}}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})](0)) = \dot{\mathbf{w}}(t) + [\nabla(\log p_{1-t}(\cdot, y))]([\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})](0))$,

(d) $\omega([\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})]'(s)) = \omega(\mathfrak{f}'(s)) + \mathbf{h}(t)$,

where we have introduced the notation

$$[\dot{\mathfrak{F}}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})](s) = \frac{d}{dt} [\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})](s)$$

and (2.6)

$$[\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})]'(s) = \frac{d}{ds} [\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})](s).$$

In fact, if $t \in [0, 1) \mapsto \Theta_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}}) \in W_2^{(m)}(\mathbb{R}; \mathbb{R}^d)$ *is defined by*

$$[\Theta_{\mathfrak{f}, y, \mathbf{h}}(0, \tilde{\mathbf{w}})](s) = \mathbf{0} \qquad \text{and} \qquad [\dot{\Theta}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})](s) = \omega([\dot{\mathfrak{F}}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})](s)), \tag{2.7}$$

then the map $t \in [0, 1) \mapsto (\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}}), \Theta_{\mathfrak{f}, \mathbf{h}}(t, \tilde{\mathbf{w}})) \in W_2^{(m)}(\mathbb{R}; \mathcal{O}(M)) \times W_2^{(m)}(\mathbb{R}; \mathbb{R}^d)$ *is uniquely determined by the system of differential equations*

$$\begin{aligned} [\dot{\mathfrak{F}}_{\mathfrak{f}, y, \mathbf{h}}(t)](s) &= \sum_{l=1}^{d} [\Theta_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})](s)_l \, \mathfrak{E}_l([\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})](s)) \\ &\qquad \text{with} \quad [\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(0, \tilde{\mathbf{w}})](s) = \mathbf{R}_{\mathcal{O}} \, \mathfrak{f}(s) \\ [\dot{\Theta}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})](s) &= [\mathcal{O}(\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}}))](s) \\ &\qquad \times (\dot{\mathbf{w}}(t) + [\nabla(p_{1-t}(\cdot, y))]([\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})](0))) \\ &\qquad + [\mathscr{A}(\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}}))](s) \, \dot{\mathbf{h}}(t) \\ &\qquad \text{with} \quad [\Theta_{\mathfrak{f}, y, \mathbf{h}}(0, \tilde{\mathbf{w}})](s) = \mathbf{0}. \end{aligned} \tag{2.8}$$

Finally,

$$\begin{aligned} \phi([\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})]'(s)) = \phi(\mathfrak{f}'(s)) + \int_0^t \sum_{l, l'=1}^{d} [\dot{\Theta}_{\mathfrak{f}, y, \mathbf{h}}(\tau, \tilde{\mathbf{w}})](s)_l \\ \times \mathbf{h}(\tau)_{l'} \, \Phi_{l, l'}([\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(\tau, \tilde{\mathbf{w}})](s)) \, d\tau, \end{aligned} \tag{2.9}$$

where

$$\Phi_{l, l'}(\mathfrak{e}) \equiv \Phi(\mathfrak{E}_l(\mathfrak{e}), \mathfrak{E}_{l'}(\mathfrak{e})), \qquad 1 \leqslant l, l' \leqslant d \qquad \text{and} \qquad \mathfrak{e} \in \mathcal{O}(M),$$

and Φ: $T(\mathcal{O}(M))^2 \to \mathfrak{o}(d)$ *is the Riemannian curvature 2-form (i.e.,* Φ *is the horizontal part of* $d\phi$*).*

Proceeding in precisely the same way as we did in the derivation of Theorem 3.1 in [ES] (in particular, employing an Euler approximation scheme to solve Stratonovich stochastic differential equations), we pass from the deterministic statements above to their stochastic analogues in the following.

2.10. THEOREM. *Let* $\mathfrak{e} \in \mathcal{O}(M)$, $y \in M$, *and* $\mathbf{h} \in \mathbf{H}$ *be given. Then, there exists a* $\tilde{\mu}$*-almost surely unique,* $\{\mathscr{F}_t : t \in [0, 1)\}$*-progressively measurable map*

$$t \in [0, 1) \mapsto (F_{\mathfrak{e}, y, \mathbf{h}}(t, \tilde{\mathbf{w}}), \Theta_{\mathfrak{e}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})) \in \bigcap_{m=1}^{\infty} W_2^{(m)}(\mathbb{R}; \mathcal{O}(M)) \times W_2^{(m)}(\mathbb{R}; \mathbb{R}^d)$$

satisfying the Stratonovich stochastic differential equations

$$\begin{aligned} d_t[\mathfrak{F}_{(\mathfrak{e},y,\mathbf{h})}(t,\tilde{\mathbf{w}})](s) &= \sum_{k=1}^{d} \mathfrak{E}_k([\mathfrak{F}_{(\mathfrak{e},y,\mathbf{h})}(t,\tilde{\mathbf{w}})](s)) \circ d_t[\Theta_{\mathfrak{e},y,\mathbf{h}}(t,\tilde{\mathbf{w}})](s) \\ &\quad \textit{with} \quad [\mathfrak{F}_{(\mathfrak{e},y,\mathbf{h})}(0,\tilde{\mathbf{w}})](s) = \mathbf{R}_{\mathcal{O}}\,\mathfrak{e} \\ d_t[\Theta_{\mathfrak{e},y,\mathbf{h}}(t,\tilde{\mathbf{w}})](s) &= [\mathcal{O}(\mathfrak{F}_{(\mathfrak{e},y,\mathbf{h})}(t,\mathbf{w}))](s) \circ d\mathbf{w}(t) \\ &\quad + ([\mathcal{O}(\mathfrak{F}_{(\mathfrak{e},y,\mathbf{h})}(t,\tilde{\mathbf{w}}))](s)[\nabla(p_{1-t}(\cdot,y))] \\ &\quad \times ([\mathfrak{F}_{(\mathfrak{e},y,\mathbf{h})}(t,\tilde{\mathbf{w}})](0)) \\ &\quad + [\mathscr{A}(\mathfrak{F}_{(\mathfrak{e},y,\mathbf{h})}(t,\tilde{\mathbf{w}}))](s)\,\dot{\mathbf{h}}(t))\,dt \\ &\quad \textit{with} \quad [\Theta_{(\mathfrak{e},y,\mathbf{h})}(0,\tilde{\mathbf{w}})](s) = \mathbf{0} \end{aligned} \tag{2.11}$$

for $\tilde{\mu}$-almost every $\tilde{\mathbf{w}} = (\mathcal{O}, \mathbf{w}) \in \mathfrak{W}$. Moreover,

$$\omega([\mathfrak{F}_{(\mathfrak{e},y,\mathbf{h})}(t,\tilde{\mathbf{w}})]'(s)) = \mathbf{h}(t)$$

and

$$\begin{aligned} &\phi([\mathfrak{F}_{(\mathfrak{e},y,\mathbf{h})}(t,\tilde{\mathbf{w}})]'(s)) \\ &\quad = \sum_{l,l'=1}^{d} \int_0^t \mathbf{h}(\tau)_{l'}\, \Phi_{l,l'}([\mathfrak{F}_{(\mathfrak{e},y,\mathbf{h})}(\tau,\tilde{\mathbf{w}})](s)) \circ d_\tau[\Theta_{(\mathfrak{e},y,\mathbf{h})}(\tau,\tilde{\mathbf{w}})](s)_l \end{aligned} \tag{2.12}$$

for $\tilde{\mu}$-almost every $\tilde{\mathbf{w}} \in \mathfrak{W}$. Finally, for each $T \in [0, 1)$, $R \in (0, \infty)$, $m \geqslant 1$, and $p \in [2, \infty)$:

$$\begin{aligned} \Bigg\| \sup_{\substack{(\mathfrak{e},y) \in \mathcal{O}(M) \times M \\ \|\mathbf{h}\|_{\mathbf{H}} \leqslant R}} \sup_{t \in [0,T]} &\|\mathfrak{F}_{(\mathfrak{e},y,\mathbf{h})}(t,\tilde{\mathbf{w}})\|_{W_2^{(m)}(\mathbb{R};\mathcal{O}(M))} \\ &\vee \|\Theta_{(\mathfrak{e},y,\mathbf{h})}(t,\tilde{\mathbf{w}})\|_{W_2^{(m)}(\mathbb{R};\mathbb{R}^d)} \Bigg\|_{L^p(\tilde{\mu})} < \infty. \end{aligned} \tag{2.13}$$

Our next goal is show that, when $\mathbf{h} \in \mathbf{H}_0 \equiv \{\mathbf{h} \in \mathbf{H} : \mathbf{h}(1) = \mathbf{0}\}$, for each $s \in \mathbb{R}$, $[\mathfrak{F}_{(\mathfrak{e},y,\mathbf{h})}(\cdot,\tilde{\mathbf{w}})](s)$ extends as a continuous function on $[0, 1]$ and that the distribution of the resulting process is absolutely continuous with respect to $\mathfrak{P}_{(x,y)}$. There are several ingredients which we require, the first of which is provided by the next lemma.

2.14. LEMMA. *Given $(\mathfrak{e}, y, \mathbf{h}) \in \mathcal{O}(M) \times M \times \mathbf{H}$, define*

$$t \in [0,1) \times \mathfrak{W} \mapsto B_{(\mathfrak{e},y,\mathbf{h})}(t,\tilde{\mathbf{w}}) \in \bigcap_{m=1}^{\infty} W_2^{(m)}(\mathbb{R};\mathbb{R}^d)$$

so that

$$[B_{(\mathfrak{e},y,\mathbf{h})}(t,\tilde{\mathbf{w}})](s) = \int_0^t [\mathcal{O}(\mathfrak{F}_{(\mathfrak{e},y,\mathbf{h})}(\tau,\tilde{\mathbf{w}}))](s)\,d\mathbf{w}(\tau), \tag{2.15}$$

where the stochastic integral in (2.15) *is taken in the sense of Itô. Then, for each* $s \in \mathbb{R}$ *and* $\tilde{\mu}$*-almost every* $\tilde{\mathbf{w}} \in \tilde{\mathfrak{W}}$,

$$[B_{(\mathfrak{e},\, y, \mathbf{h})}(1, \tilde{\mathbf{w}})](s) \equiv \lim_{t \nearrow 1} [B_{\mathfrak{e},\, y, \mathbf{h}}(t, \tilde{\mathbf{w}})](s) \text{ exists,}$$

and $\tilde{\mathbf{w}} \rightsquigarrow [B_{(\mathfrak{e},\, y, \mathbf{h})}(\cdot, \tilde{\mathbf{w}})](s)$ *has the same distribution under* $\tilde{\mu}$ *as* $\tilde{\mathbf{w}} \rightsquigarrow \mathbf{w}$ *does. Furthermore, if* $\mathrm{Ric}\colon \mathcal{O}(M) \to \mathbb{R}^d \otimes \mathbb{R}^d$ *is the Ricci curvature (symmetric) matrix, given by*

$$(\mathbf{v}, \mathrm{Ric}(\mathfrak{e})\mathbf{v}')_{\mathbb{R}^d} = \sum_{k=1}^{d} (\Phi(\mathfrak{E}(\mathbf{v})_{\mathfrak{e}}, \mathfrak{E}_k(\mathfrak{e}))\, \mathbf{e}_k, \mathbf{v}')_{\mathbb{R}^d}, \qquad \mathbf{v}, \mathbf{v}' \in \mathbb{R}^d, \quad (2.16)$$

and $\Xi\colon W_2^{(m)}(\mathbb{R}; \mathcal{O}(M)) \to W_2^{(m)}(\mathbb{R}; \mathbb{R}^d)$ *is defined by*

$$[\Xi(\mathfrak{f})](s) = \int_0^s [\mathcal{O}(\mathfrak{f})](\sigma)^{\top} \mathrm{Ric}(\mathfrak{f}(\sigma))\, \omega(\mathfrak{f}'(\sigma))\, d\sigma, \qquad (2.17)$$

then, for each $s \in \mathbb{R}$ *and* $\tilde{\mu}$*-almost every* $\tilde{\mathbf{w}}$,

$$\int_0^t [\mathcal{O}(\mathfrak{F}_{(\mathfrak{e},\, y, \mathbf{h})}(\tau, \tilde{\mathbf{w}})](\sigma) \circ d\mathbf{w}(\tau)$$

$$= [B_{(\mathfrak{e},\, y, \mathbf{h})}(t, \tilde{\mathbf{w}})](s) + \tfrac{1}{2} \int_0^t [\mathcal{O}\Xi(\mathfrak{F}_{(\mathfrak{e},\, y, \mathbf{h})}(\tau, \tilde{\mathbf{w}}))](s)\, d\tau, \qquad t \in [0, 1). \qquad (2.18)$$

Proof. Because $\mathfrak{F}_{(\mathfrak{e},\, y, \mathbf{h})}(\cdot, \tilde{\mathbf{w}})$ is progressively measurable and $[\mathcal{O}(\mathfrak{f})](s) \in \mathcal{O}(d)$, everything except (2.18) is a completely standard consequence of Itô integration theory. In addition, (2.18) is just an expression of the formula by which one passes from Stratonovich integrals to Itô integrals. The only difficulty here is that we are dealing with semimartingales which take their values in the path space $W_2^{(1)}(\mathbb{R}; \mathcal{O}(M))$, which is not a Hilbert space. However, this objection can be easily overcome by embedding $\mathcal{O}(M)$ in $\mathbb{R}^N$ for some sufficiently large N and then realizing $W_2^{(1)}(\mathbb{R}\mathcal{O}(M))$ as a closed subspace of $W^{(1)}(\mathbb{R}; \mathbb{R}^N)$. Moreover, when one does this and one uses the Hilbert structure of $W_2^{(1)}(\mathbb{R}; \mathbb{R}^N)$ to carry out the computation in terms of an orthonormal basis, one finds that the required Stratonovich correction is given by

$$\tfrac{1}{2} \sum_{k=1}^{d} \int_0^t [\mathcal{S}_k \mathcal{O}(\mathfrak{F}_{(\mathfrak{e},\, y, \mathbf{h})}(\tau, \tilde{\mathbf{w}}))](s)\, \mathbf{e}_k\, d\tau,$$

where the operator $\mathscr{S}_k$ is defined on $F \in C^1(W_2^{(1)}(\mathbb{R}; \mathcal{O}(M)); \mathbb{R})$ so that

$$[\mathscr{S}_k F(\mathfrak{f})](s) = \frac{d}{d\varepsilon} F(\mathfrak{f}_k(\varepsilon))\bigg|_{\varepsilon=0}$$

and $\varepsilon \in \mathbb{R} \mapsto \mathfrak{f}_k \in W_2^{(1)}(\mathbb{R}; \mathcal{O}(M))$ is determined by

$$\frac{d}{d\varepsilon}[\mathfrak{f}_k(\varepsilon)](s) = \mathfrak{E}([\mathcal{O}(\mathfrak{f})](s)\mathbf{e}_k)_{[\mathfrak{f}_k(\varepsilon)](s)} \qquad \text{with} \quad \mathfrak{f}_k(0) = \mathfrak{f}.$$

Hence, everything comes down to computing $\sum_1^d \mathscr{S}_k \mathcal{O} \mathbf{e}_k$.

Let $1 \leqslant k \leqslant d$ and $\mathfrak{f}$ be given. Starting from (2.3), it is an elementary step to first

$$[\mathscr{S}_k \mathcal{O}(\mathfrak{f})]'(s) + \phi(\mathfrak{f}(s))[\mathscr{S}_k \mathcal{O}(\mathfrak{f})](s) = -\mathscr{S}_k(\phi \circ \mathfrak{f}'(s))[\mathcal{O}(\mathfrak{f})](s)$$

and then to

$$[\mathscr{S}_k \mathcal{O}(\mathfrak{f})](s) = -[\mathcal{O}(\mathfrak{f})](s) \int_0^s [\mathcal{O}(\mathfrak{f})](\sigma)^\top \mathscr{S}_k(\phi \circ \mathfrak{f}(s))[\mathcal{O}(\mathfrak{f})](\sigma)\, d\sigma.$$

Thus, all that remains is to check that

$$\sum_1^d \mathscr{S}_k(\phi \circ \mathfrak{f}(s))[\mathcal{O}(\mathfrak{f})](s)\mathbf{e}_k = -\mathrm{Ric}(\mathfrak{f}(s))\, \omega(\mathfrak{f}'(s)).$$

But, because $\mathfrak{E}$ is always horizontal, the second structural equation leads to

$$\begin{aligned}
\frac{d}{d\varepsilon}\phi([\mathfrak{f}_k'(\varepsilon)](s))\bigg|_{\varepsilon=0} &= d\phi(\mathfrak{E}([\mathcal{O}(\mathfrak{f})](s)\mathbf{e}_k)_{\mathfrak{f}(s)}, \mathfrak{f}'(s)) \\
&= \Phi(\mathfrak{E}([\mathcal{O}(\mathfrak{f})](s)\mathbf{e}_k)_{\mathfrak{f}(s)}, \mathfrak{f}'(s)) \\
&= -\Phi(\mathfrak{f}'(s), \mathfrak{E}([\mathcal{O}(\mathfrak{f})](s)\mathbf{e}_k)_{\mathfrak{f}(s)}),
\end{aligned}$$

and so

$$\begin{aligned}
&\sum_1^d \mathscr{S}_k(\phi \circ \mathfrak{f}(s))[\mathcal{O}(\mathfrak{f})](s)\mathbf{e}_k \\
&\qquad = -\sum_1^d \Phi(\mathfrak{f}'(s), \mathfrak{E}([\mathcal{O}(\mathfrak{f})](s)\mathbf{e}_k)_{\mathfrak{f}(s)})[\mathcal{O}(\mathfrak{f})](s)\mathbf{e}_k,
\end{aligned}$$

from which the desired conclusion is an easy step. ∎

In order to carry out the next step, we will need to know that, for $\mathfrak{f} \in C^1(\mathbb{R}; \mathcal{O}(M))$ (cf. (1.3) and (2.3)):

$$\begin{aligned}&\nabla\varphi(\mathfrak{f}(s)) - [\mathcal{O}(\mathfrak{f})](s)\,\nabla\varphi(\mathfrak{f}(0))\\&\quad= [\mathcal{O}(\mathfrak{f})](s) \int_0^s [\mathcal{O}(\mathfrak{f})](\sigma)^\top\, [\mathrm{Hess}(\varphi)](\mathfrak{f}(\sigma))\, \omega(\mathfrak{f}'(\sigma))\, d\sigma,\\&\quad \varphi \in C^2(M; \mathbb{R}),\end{aligned} \tag{2.19}$$

where the Hessian $\mathrm{Hess}(\varphi)\colon \mathcal{O}(M) \to \mathbb{R}^d \otimes \mathbb{R}^d$ is the symmetric matrix valued function given by

$$[\mathrm{Hess}(\varphi)](\mathfrak{e}) = ((\mathfrak{E}_i \mathfrak{E}_j \varphi \circ \pi(\mathfrak{e})))_{i \leqslant i, j \leqslant d}. \tag{2.20}$$

To check (2.19), set $\mathcal{O}(s) = [\mathcal{O}(\mathfrak{f})](s)$, and define $\bar{\mathfrak{f}} \in C^1(\mathbb{R}; \mathcal{O}(M))$ by

$$\bar{\mathfrak{f}}'(s) = \mathfrak{E}([\mathcal{O}(\mathfrak{f})](s)\, \omega(\mathfrak{f}'(s)))_{\bar{\mathfrak{f}}(s)} \qquad \text{with} \quad \bar{\mathfrak{f}}(0) = \mathfrak{f}(0).$$

Then, after comparing their first derivatives, one sees that $\mathfrak{f}(s) = R_{\mathcal{O}(s)^\top} \bar{\mathfrak{f}}(s)$, and, therefore, $\nabla\varphi(\mathfrak{f}(s)) = \mathcal{O}(s)^\top\, \nabla\varphi(\bar{\mathfrak{f}}(s))$. Hence, the left side of (2.19) is equal

$$\mathcal{O}(s)(\nabla\varphi(\bar{\mathfrak{f}}(s)) - \nabla\varphi(\bar{\mathfrak{f}}(0))) = \mathcal{O}(s) \int_0^s [\mathrm{Hess}(\varphi)](\bar{\mathfrak{f}}(\sigma))\, \omega(\bar{\mathfrak{f}}'(\sigma))\, d\sigma.$$

Finally, (2.19) follows from this combined with

$$[\mathrm{Hess}(\varphi)](\mathbf{R}_{\mathcal{O}}\, \mathfrak{e}) = \mathcal{O}^\top [\mathrm{Hess}(\varphi)](\mathfrak{e})\, \mathcal{O}.$$

In the following, and thereafter, when F is a function on $\mathbb{R}$ and $s \in \mathbb{R}$, we use F_s to denote the shifted function $s' \rightsquigarrow F(s + s')$.

2.21. Lemma. *Given* $(\mathfrak{e}, y, \mathbf{h}) \in M \times \mathbf{H}$, $(t, \tilde{\mathbf{w}}) \in [0, 1) \times \mathfrak{W}$, *and* $s \in \mathbb{R}$, *set*

$$\widetilde{\mathrm{Ric}}(t, \mathfrak{e}, y) = \tfrac{1}{2}\,\mathrm{Ric}(\mathfrak{e}) - H(t, \mathfrak{e}, y)$$

$$\textit{where} \quad H(t, \cdot, y) \equiv \mathrm{Hess}(\log p_{1-t}(\cdot, y)),$$

$$\begin{aligned}[\beta_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})](s) &= \int_0^s [\mathcal{O}(\mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}}))](\sigma)^\top\\&\quad \times (\dot{\mathbf{h}}(t) + \widetilde{\mathrm{Ric}}(t, [\mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})](\sigma), y)\, \mathbf{h}(t))\, d\sigma,\\ [\hat{B}_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})](s) &= [B_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})](s)\\&\quad + \int_0^t [\mathcal{O}(\mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(\tau, \tilde{\mathbf{w}}))](s) [\beta_{(\mathfrak{e}, y, \mathbf{h})}(\tau, \tilde{\mathbf{w}})](s)\, d\tau.\end{aligned} \tag{2.22}$$

Then, for $\tilde{\mu}$-almost every $\tilde{\mathbf{w}} \in \tilde{\mathfrak{W}}$,

$$\begin{aligned} d_t[\Theta_{\mathfrak{e}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})_s](s') &= [\mathcal{O}(\mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})_s](s') \circ d_t[\hat{B}_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})](s) \\ &\quad + ([\mathcal{O}(\mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})_s)](s')[\nabla(p_{1-t}(\cdot, y))]([\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})_s](0)) \\ &\quad + [\mathscr{A}(\mathfrak{F}_{\mathfrak{f}, y, \mathbf{h}}(t, \tilde{\mathbf{w}})_s)](s')\, \dot{\mathbf{h}}(t))\, dt. \end{aligned} \tag{2.23}$$

Proof. Given (2.11), (2.18) and (2.19) (applied with $\varphi = p_{1-t}(\cdot, y)$ and $\mathfrak{f} = \mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})$) the verifaction of (2.23) reduces to the observation that

$$\begin{aligned} &[\mathcal{O}(\mathfrak{f})](s+s') = [\mathcal{O}(\mathfrak{f}_s)](s')[\mathcal{O}(\mathfrak{f})](s) \\ \text{and } &[\mathscr{A}(\mathfrak{f})](s+s') = [\mathscr{A}(\mathfrak{f}_s)](s') + [\mathcal{O}(\mathfrak{f}_s)](s')[\mathscr{A}(\mathfrak{f})](s). \quad \blacksquare \end{aligned} \tag{2.24}$$

Note that, because it is obvious that

$$\begin{aligned} d_t[\mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})_s](s') &= \sum_{k=1}^{d} \mathfrak{E}_k([\mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})s](s')) \circ d_t[\Theta_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})_s](s'). \end{aligned} \tag{2.25}$$

Equation (2.23) is saying that $\mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(\cdot, \tilde{\mathbf{w}})_s$ bears the same relation to $[\hat{B}_{(\mathfrak{e}, y, \mathbf{h})}(\cdot, \tilde{\mathbf{w}})](s)$ as $\mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(\cdot, \tilde{\mathbf{w}})$ does to $\mathbf{w}$. Moreover, $[\hat{B}_{(\mathfrak{e}, y, \mathbf{h})}(\cdot, \tilde{\mathbf{w}})](s)$ differs from $[B_{(\mathfrak{e}, y, \mathbf{h})}(\cdot, \tilde{\mathbf{w}})](s)$ by a function which is absolutely continuous, and $[B_{(\mathfrak{e}, y, \mathbf{h})}(\cdot, \tilde{\mathbf{w}})](s)$ is a Brownian motion. Thus, what we want to do is *remove* this absolutely continuous difference by using the Cameron–Martin–Girsanov transformation formula. However, there is a technical problem here which arises from the fact that $H(t, \mathfrak{e}, y)$ has a singularity at $t = 1$. For this reason, we will need the next result.

2.26. LEMMA. *There is a $C \in (0, \infty)$ with the properties that (cf.* (1.30) *and* (2.22)), *$\tilde{\mu}$-almost surely,*

$$\begin{aligned} &|[\beta_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})]'(s)|^2 \\ &\quad \leqslant C\left(|\dot{\mathbf{h}}(t)|^2 + \frac{|\mathbf{h}(t)|^2}{(1-t)^2} + \frac{s^4\, |\mathbf{h}(t)|^6 + \operatorname{dist}(\pi \circ \mathfrak{F}_{(\mathfrak{e}, y)}(t,. \tilde{\mathbf{w}}), y)^4\, |\mathbf{h}(t)|^2}{(1-t)^4}\right) \end{aligned} \tag{2.27}$$

and

$$\mathbb{E}^{\tilde{\mu}}[\operatorname{dist}(\pi \circ \mathfrak{F}_{(\mathfrak{e}, y)}(t, \tilde{\mathbf{w}}), y)^4] \leqslant C(1-t)^2. \tag{2.28}$$

In particular, there is a $C \in (0, \infty)$ *such that*

$$\int_0^1 |[\beta_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})]'(s)|^2 \, dt$$
$$\leqslant C\left(\|\mathbf{h}\|_{\mathbf{H}}^2 + s^4 \|\mathbf{h}\|_{\mathbf{H}}^6 + \int_0^1 \frac{\operatorname{dist}(\pi \circ \mathfrak{F}_{(\mathfrak{e}, y)}(t, \tilde{\mathbf{w}}), y)^4 \, |\mathbf{h}(t)|^2}{(1-t)^4} \, dt \right), \quad (2.29)$$

and, for each $L \in (0, \infty)$,

$$\sup_{\substack{h \in \mathbf{H}_0 \\ \|\mathbf{h}\|_{\mathbf{H}} \leqslant L}} \mathbb{E}^{\tilde{\mu}} \left[\int_0^1 \frac{\operatorname{dist}(\pi \circ \mathfrak{F}_{(\mathfrak{e}, y)}(t, \tilde{\mathbf{w}}), y)^4 \, |\mathbf{h}(t)|^2}{(1-t)^4} \, dt \right] \leqslant CL^2; \qquad (2.30)$$

and so,

$$\sup_{(\mathfrak{e}, y) \in \mathcal{O}(M) \times M} \; \sup_{\substack{\mathbf{h} \in \mathbf{H}_0 \\ \|\mathbf{h}\|_{\mathbf{H}} \leqslant L}} \mathbb{E}^{\tilde{\mu}} \left[\sup_{|s| \leqslant L} \int_0^1 |[\beta_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})]'(s)|^2 \, dt \right] < \infty. \qquad (2.31)$$

Proof. Clearly, (2.27) reduces to showing that (cf. 2.22)) there is a $C < \infty$ such that

$$\|H(t, [\mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})](s), y, \mathbf{h})\|_{\mathrm{op}}$$
$$\leqslant C\left(\frac{1}{1-t} + \frac{s^2 \, |\mathbf{h}(t)|^2 + \operatorname{dist}(\pi \circ \mathfrak{F}_{(\mathfrak{e}, y)}(t, \tilde{\mathbf{w}}), y)^2}{(1-t)^2} \right);$$

and, in view of (0.3) in [S], this comes down to the observation that

$$\operatorname{dist}(\pi \circ \mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})](s), y) \leqslant s \, |\mathbf{h}(t)| + \operatorname{dist}(\pi \circ \mathfrak{F}_{(\mathfrak{e}, y)}(t, \tilde{\mathbf{w}}), y), \quad (2.32)$$

since, by uniqueness for solutions to (2.11), $[\mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(\cdot, \tilde{\mathbf{w}})](0) = \mathfrak{F}_{(\mathfrak{e}, y)}(\cdot, \tilde{\mathbf{w}})$ $\tilde{\mu}$-almost surely and, by the first equation in (2.12),

$$\operatorname{dist}([\pi \circ \mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})](s), \pi \circ [\mathfrak{F}_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})](0)) \leqslant s \, |\mathbf{h}(t)|.$$

To prove (2.28), set $x = \pi(\mathfrak{e})$, and recall (cf. Corollary 1.29, (1.20), and (1.21)) that

$$\mathbb{E}^{\tilde{\mu}}[\operatorname{dist}(\pi \circ \mathfrak{F}_{(\mathfrak{e}, y)}(t, \tilde{\mathbf{w}}), y)^4] = \mathbb{E}^{\mathfrak{P}_{(y,x)}}[\operatorname{dist}(\pi \circ \mathfrak{p}(1-t), y)^4]$$
$$= \mathbb{E}^{P_y} \left[\operatorname{dist}(p(1-t), y)^4 \, \frac{p_{1/2}(p(\frac{1}{2}), x)}{p_1(y, x)} \right]$$

for $t \in [\frac{1}{2}, 1)$. Hence, (2.28) is an easy application of standard estimates about the rate at which a diffusion leaves a point.

Given (2.27) and (2.28), the proofs of (2.29) and (2.30) are easy applications of Hardy's inequality, which says that

$$\int_0^\infty \left(\frac{1}{t}\int_0^t f(\tau)\,d\tau\right)^2 dt \leqslant 4\int_0^\infty |f(t)|^2\,dt. \tag{2.33}$$

Indeed, if $\mathbf{h}\in\mathbf{H}_0$ and we apply (2.33) with $f(t)=\mathbf{1}_{[0,1]}(t)\,|\dot{\mathbf{h}}(1-t)|$, then we see that

$$\int_0^1 \frac{|\mathbf{h}(t)|^2}{(1-t)^2}\,dt \leqslant 4\,\|\mathbf{h}\|_{\mathbf{H}}^2,$$

and, because $|\mathbf{h}(1-t)|^4 \leqslant (1-t)^2\,\|\mathbf{h}\|_{\mathbf{H}}^4$,

$$\int_0^1 \frac{|\mathbf{h}(t)|^6}{(1-t)^4}\,dt \leqslant 4\,\|\mathbf{h}\|_{\mathbf{H}}^6,$$

which proves (2.29). Finally, because of (2.28), the proof of (2.30) is similar. ∎

As a consequence of (2.31), we know that, for each $s\in\mathbb{R}$, and $(\mathfrak{e}, y, \mathbf{h})\in\mathcal{O}(M)\times M\times\mathbf{H}_0$, the Itô stochastic integral

$$(t, \tilde{\mathbf{w}})\in[0,1]\times\mathfrak{W}\mapsto\int_0^t ([\beta_{(\mathfrak{e}, y, \mathbf{h})}(\tau, \tilde{\mathbf{w}})](s), d\mathbf{w}(\tau))_{\mathbb{R}^d}\in\mathbb{R}$$

is $\tilde{\mu}$-almost surely well-defined and determines a progressively measurable function which is $\tilde{\mu}$-almost surely continuous with respect to $t\in[0,1]$. Moreover, if

$$\begin{aligned}[\rho_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})](s) = &\int_0^t ([\beta_{(\mathfrak{e}, y, \mathbf{h})}(\tau, \tilde{\mathbf{w}})](s), d\mathbf{w}(\tau))_{\mathbb{R}^d}\\ &+\tfrac{1}{2}\int_0^1 |[\beta_{(\mathfrak{e}, y, \mathbf{h})}(\tau, \tilde{\mathbf{w}})](s)|^2\,d\tau\end{aligned} \tag{2.34}$$

and

$$[\mathbf{R}_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})](s) = \exp[-\rho_{(\mathfrak{e}, y, \mathbf{h})}(t, \tilde{\mathbf{w}})], \tag{2.35}$$

then $([\mathbf{R}_{(\mathfrak{e}, y, \mathbf{h})}(t)](s), \mathcal{F}_t, \tilde{\mu})$, $t\in[0,1)$, is a positive martingale with mean-value 1.

2.36. THEOREM. *Let $(\mathfrak{e}, y, \mathbf{h})\in\mathcal{O}(M)\times M\times\mathbf{H}_0$ be given. For each $s\in\mathbb{R}$,*

$$[\mathbf{R}_{(\mathfrak{e}, y, \mathbf{h})}(t)](s)\to[\mathbf{R}_{(\mathfrak{e}, y, \mathbf{h})}(1)](s) \qquad \text{as} \quad t\nearrow 1,$$
$$\text{both } \tilde{\mu}\text{-almost surely and in } L^1(\tilde{\mu}). \tag{2.37}$$

In particular, $(\mathbf{R}_{(e,\,y,\mathbf{h})}(t), \mathscr{F}_t, \tilde{\mu})$ *extends to* $[0, 1]$ *as a positve,* $\tilde{\mu}$*-almost surely continuous martingale with mean-value* 1. *Finally, if* $\tilde{\mu}^s_{(e,\,y,\mathbf{h})}$ *is the probability measure on* $\tilde{\mathfrak{W}}$ *given by*

$$\tilde{\mu}^s_{(e,\,y,\mathbf{h})}(d\tilde{\mathbf{w}}) = [\mathbf{R}_{(e,\,y,\mathbf{h})}(1, \tilde{\mathbf{w}})](s)\, \tilde{\mu}(d\tilde{\mathbf{w}}), \tag{2.38}$$

then (*cf. the notation introduced just before Lemma* 2.21)

$$\begin{aligned}(\mathfrak{F}_{(e,\,y,\mathbf{h})}(\cdot, \tilde{\mathbf{w}})_s, [\hat{B}_{(e,\,y,\mathbf{h})}(\cdot, \tilde{\mathbf{w}})](s)) \qquad &\textit{under} \quad \tilde{\mu}^s_{(e,\,y,\mathbf{h})} \\ \overset{\text{law}}{=} (\mathfrak{F}_{(e,\,y,\mathbf{h})}(\cdot, \tilde{\mathbf{w}}), \tilde{\mathbf{w}}) \qquad &\textit{under } \tilde{\mu}.\end{aligned} \tag{2.39}$$

Proof. The a.s. convergence in (2.37) is immediate, and so the first part comes down to checking that the convergence is also in $L'(\tilde{\mu})$. To this end, set

$$\begin{aligned}\zeta_L(\tilde{\mathbf{w}}) &= \left\{ t \in [0, 1] : \int_0^1 \frac{\operatorname{dist}(\pi \circ \mathfrak{F}_{(e,\,y)}(\tau, \tilde{\mathbf{w}}), y)^4\, |\mathbf{h}(\tau)|^2}{(1-\tau)^4}\, d\tau \geqslant L \right\} \\ &\equiv 1 \qquad \text{if} \quad \int_0^1 \frac{\operatorname{dist}(\pi \circ \mathfrak{F}_{(e,\,y)}(\tau, \tilde{\mathbf{w}}), y)^4\, |\mathbf{h}(\tau)|^2}{(1-\tau)^4}\, d\tau < L,\end{aligned}$$

and put $\mathbf{R}_L(s, \tilde{\mathbf{w}}) = [\mathbf{R}_{(e,\,y,\mathbf{h})}(\zeta_L(\tilde{\mathbf{w}}), \tilde{\mathbf{w}})](s)$. By (2.29), $\mathbb{E}^{\tilde{\mu}}[\mathbf{R}_L(s)] = 1$ for every $s \in \mathbb{R}$. Thus, if $d\tilde{\mu}^s_L = \mathbf{R}_L(s)\, d\tilde{\mu}$, then $\tilde{\mu}^s_L$ is a Borel probability measure on $\tilde{\mathfrak{W}}$, and, by the Cameron–Martin–Girsanov transformation formula,

$$\begin{aligned}[\hat{B}_{(e,\,y,\mathbf{h})}(\cdot \wedge \zeta_L(\tilde{\mathbf{w}}), \tilde{\mathbf{w}})](s) \qquad &\text{under} \quad \tilde{\mu}^s_L \\ \overset{\text{law}}{=} [B_{(e,\,y,\mathbf{h})}(\cdot \wedge \zeta_L(\tilde{\mathbf{w}}), \tilde{\mathbf{w}})](s) \qquad &\text{under } \tilde{\mu}.\end{aligned}$$

But (cf. the comment following Lemma 2.21) this means that

$$\begin{aligned}(\mathfrak{F}_{(e,\,y,\mathbf{h})}(\cdot \wedge \zeta_L(\tilde{\mathbf{w}}), \tilde{\mathbf{w}})_s, [\hat{B}_{(e,\,y,\mathbf{h})}(\cdot \wedge \zeta_L(\tilde{\mathbf{w}}), \tilde{\mathbf{w}})](s)) \qquad &\text{under} \quad \tilde{\mu}^s_L \\ \overset{\text{law}}{=} (\mathfrak{F}_{(e,\,y,\mathbf{h})}(\cdot \wedge \zeta_L(\tilde{\mathbf{w}}), \tilde{\mathbf{w}})](s)) \qquad &\text{under } \tilde{\mu}.\end{aligned} \tag{2.40}$$

In particular,

$$\begin{aligned}&\mathbb{E}^{\tilde{\mu}}[\,|[\mathbf{R}_{(e,\,y,\mathbf{h})}(1)](s) - \mathbf{R}_L(s)|\,] \\ &\quad \leqslant \mathbb{E}^{\tilde{\mu}}[[\mathbf{R}_{e,\,y,\mathbf{h}}(1)](s), \zeta_L < 1] + \mathbb{E}^{\tilde{\mu}}[\mathbf{R}_L(s), \zeta_L < 1] \\ &\quad \leqslant 2\mathbb{E}^{\tilde{\mu}}[\mathbf{R}_L(s), \zeta_L < 1] \\ &\quad \leqslant 2\tilde{\mu}\left(\int_0^1 \frac{\operatorname{dist}(\pi \circ [\mathfrak{F}_{(e,\,y,\mathbf{h})}(t)](-s), y)^4\, |\mathbf{h}(t)|^2}{(1-t)^4}\, dt \geqslant L \right),\end{aligned}$$

where we have used the trivial identity

$$[\mathfrak{F}_{(e,\,y,\mathbf{h})}(t,\tilde{\mathbf{w}})_s](-s)=\mathfrak{F}_{(e,\,y)}(t,\tilde{\mathbf{w}}) \qquad (\text{a.s.}, \tilde{\mu})$$

plus (2.40) in the justification of the last inequality. Hence, by (2.32) and (2.30), we conclude that, for any $S\in(0,\infty)$,

$$\lim_{L\to\infty}\ \sup_{|s|\leqslant S}\ \mathbb{E}^{\tilde{\mu}}[|\mathbf{R}_L(s)-[\mathbf{R}_{(e,\,y,\mathbf{h})}(1)](s)|]=0. \tag{2.41}$$

Finally, (2.41) is more than enough to assure the desired extension of $[\mathbf{R}_{(e,\,y,\mathbf{h})}(\cdot)](s)$ to $[0,1]$. In addition, once one knows that this extension exists, (2.39) becomes an easy consequence of (2.40). ∎

2.42. Corollary. *For each* $(e,y,\mathbf{h})\in\mathcal{O}(M)\times M\times\mathbf{H}_0$,

$$\lim_{s\to0} s^{-1}\mathbb{E}^{\tilde{\mu}}\Bigg[\Bigg|[\mathbf{R}_{(e,\,y,\mathbf{h})}(1,\tilde{\mathbf{w}})](s)-1+ \\ +s\int_0^1(\dot{\mathbf{h}}(t)+\widetilde{\mathrm{Ric}}(t,\mathfrak{F}_{(e,\,y)}(t,\tilde{\mathbf{w}}),y)\,\mathbf{h}(t),d\mathbf{w}(t))_{\mathbb{R}^d}\Bigg|\Bigg]=0. \tag{2.43}$$

Proof. Notice that (cf. (2.34) and (2.35))

$$[\mathbf{R}_{(e,\,y,\mathbf{h})}(t,\tilde{\mathbf{w}})]'(s)=-[\rho_{(e,\,y,\mathbf{h})}(t,\tilde{\mathbf{w}})]'(s)[\mathbf{R}_{(e,\,y,\mathbf{h})}(1,\tilde{\mathbf{w}})](s)$$

and $[\rho_{(e,\,y,\mathbf{h})}(1,\tilde{\mathbf{w}})]'(0)=\int_0^1(\dot{\mathbf{h}}(t)+\widetilde{\mathrm{Ric}}(t,\mathfrak{F}_{(e,\,y)}(t,\tilde{\mathbf{w}}),y)\,\mathbf{h}(t),d\mathbf{w}(t))_{\mathbb{R}^d}$.

Hence, for each $s\in\mathbb{R}\backslash\{0\}$, the expectation value on the left hand side of (2.43) is dominated by

$$\int_0^s\mathbb{E}^{\tilde{\mu}}[|[\rho_{(e,\,y,\mathbf{h})}(1)](\sigma)-[\rho_{(e,\,y,\mathbf{h})}(1)](0)|\,[\mathbf{R}_{(e,\,y,\mathbf{h})}(1)](\sigma)]\,d\sigma \\ +\int_0^s\mathbb{E}^{\tilde{\mu}}[|[\rho_{(e,\,y,\mathbf{h})}(1)](0)|\,|[\mathbf{R}_{(e,\,y,\mathbf{h})}(1)](\sigma)-1|]\,d\sigma. \tag{2.44}$$

By (2.39), the first term in (2.44) is equal

$$\int_0^s\mathbb{E}^{\tilde{\mu}}[|[\rho_{(e,\,y,\mathbf{h})}(1)](0)-[\rho_{(e,\,y,\mathbf{h})}(1)](-\sigma)|]\,d\sigma,$$

which, because of the estimates in (2.30) and (2.32), tends to zero faster than s. As for the second term in (2.44) dominate it by

$$\int_0^s \mathbb{E}^{\tilde{\mu}}[1+[\mathbf{R}_{(\mathfrak{e},\,y,\mathbf{h})}(1)](\sigma),\ |[\rho_{(\mathfrak{e},\,y,\mathbf{h})}(1)](0)| \geqslant L]\,d\sigma$$

$$+L\int_0^s \mathbb{E}^{\tilde{\mu}}[\,|[\mathbf{R}_{(\mathfrak{e},\,y,\mathbf{h})}(1)](\sigma)-1|\,]\,d\sigma.$$

By another application of (2.39), the first of these is dominated by

$$s\tilde{\mu}(|[\rho_{(\mathfrak{e},\,y,\mathbf{h})}(1)](0) \geqslant L)+\int_0^s \tilde{\mu}(|[\rho_{(\mathfrak{e},\,y,\mathbf{h})}(1)](-\sigma)| \geqslant L)\,d\sigma,$$

and so (2.30) and (2.32) lead to

$$\lim_{L\nearrow\infty}\ \lim_{s\to 0} s^{-1}\int_0^s \mathbb{E}^{\tilde{\mu}}[1+[\mathbf{R}_{(\mathfrak{e},\,y,\mathbf{h})}(1)](\sigma),\ |[\rho_{(\mathfrak{e},\,y,\mathbf{h})}(1)](0)| \geqslant L]\,d\sigma=0.$$

Finally, because

$$\begin{aligned}
&\mathbb{E}^{\tilde{\mu}}[\,|[\mathbf{R}_{(\mathfrak{e},\,y,\mathbf{h})}(1)](\sigma)-1|\,]\\
&\quad\leqslant \int_0^\sigma \mathbb{E}^{\tilde{\mu}}[\,|[\rho_{(\mathfrak{e},\,y,\mathbf{h})}(1)](\sigma')|\ [\mathbf{R}_{(\mathfrak{e},\,y,\mathbf{h})}(1)](\sigma')]\,d\sigma'\\
&\quad=\sigma\,\mathbb{E}^{\tilde{\mu}}[\,|[\rho_{(\mathfrak{e},\,y,\mathbf{h})}(1)](0)|\,],
\end{aligned}$$

we are done. ▮

3. Integration by Parts

We now have all the prerequisites for proving an integration by parts formula. The basic result is contained in the following.

3.1. Theorem. *Let* $(\mathfrak{e},\,y,\,\mathbf{h})\in\mathcal{O}(M)\times M\times\mathbf{H}$ *and* $F,\,G\in C_b(\mathscr{P}_1(\mathcal{O}(M));\,\mathbb{R})$ *be given. Then*

$$s\in\mathbb{R}\mapsto\mathbb{E}^{\tilde{\mu}}[F([\mathfrak{F}_{(\mathfrak{e},\,y,\mathbf{h})}(\cdot,\tilde{\mathbf{w}})](s))\ G(\mathfrak{F}_{(\mathfrak{e},\,y)}(\cdot,\tilde{\mathbf{w}}))]$$

is differentiable at $s=0$ *if and only if*

$$s\in\mathbb{R}\mapsto\mathbb{E}^{\tilde{\mu}}[F(\mathfrak{F}_{(\mathfrak{e},\,y)}(\cdot,\tilde{\mathbf{w}}))\ G([\mathfrak{F}_{(\mathfrak{e},\,y,\mathbf{h})}(\cdot,\tilde{\mathbf{w}})](s))] \tag{3.2}$$

is, in which case

$$\frac{d}{ds}\mathbb{E}^{\tilde{\mu}}[F([\mathfrak{F}_{(\mathbf{e},y,\mathbf{h})}(\cdot,\tilde{\mathbf{w}})](s))\,G(\mathfrak{F}_{(\mathbf{e},y)}(\cdot,\tilde{\mathbf{w}}))]\Big|_{s=0}$$

$$= -\frac{d}{ds}\mathbb{E}^{\tilde{\mu}}[F(\mathfrak{F}_{(\mathbf{e},y)}(\cdot,\tilde{\mathbf{w}}))\,G([\mathfrak{F}_{(\mathbf{e},y,\mathbf{h})}(\cdot,\tilde{\mathbf{w}})](s))]\Big|_{s=0}$$

$$+\mathbb{E}^{\tilde{\mu}}\left[\int_0^1 (\dot{\mathbf{h}}(t)+\widetilde{\mathrm{Ric}}(t,\mathfrak{F}_{(\mathbf{e},y)}(t,\tilde{\mathbf{w}}))\,\mathbf{h}(t),d\mathbf{w}(t))_{\mathbb{R}^d}\right.$$

$$\left.\times(FG)(\mathfrak{F}_{(\mathbf{e},y)}(\cdot,\tilde{\mathbf{w}}))\right]. \tag{3.3}$$

Proof. After a few easy manipulations and an application of (2.39), one arrives at

$$\mathbb{E}^{\tilde{\mu}}[(F([\mathfrak{F}_{(\mathbf{e},y,\mathbf{h})}(\cdot,\tilde{\mathbf{w}})](s))-F(\mathfrak{F}_{(\mathbf{e},y)}(\cdot,\tilde{\mathbf{w}})))\,G(\mathfrak{F}_{(\mathbf{e},y)}(\cdot,\tilde{\mathbf{w}}))]$$

$$=\mathbb{E}^{\tilde{\mu}}[F(\mathfrak{F}_{(\mathbf{e},y)}(\cdot,\tilde{\mathbf{w}}))\,G([\mathfrak{F}_{(\mathbf{e},y,-\mathbf{h})}(\cdot,\tilde{\mathbf{w}})](s))([\mathbf{R}_{(\mathbf{e},y,-\mathbf{h})}(1,\tilde{\mathbf{w}})](s)-1)]$$

$$+\mathbb{E}^{\tilde{\mu}}[F(\mathfrak{F}_{(\mathbf{e},y)}(\cdot,\tilde{\mathbf{w}}))(G([\mathfrak{F}_{(\mathbf{e},y,-\mathbf{h})}(\cdot,\tilde{\mathbf{w}})](s))-G(\mathfrak{F}_{(\mathbf{e},y)}(\cdot,\tilde{\mathbf{w}})))],$$

from which (3.3) follows after an obvious limiting procedure which is justified by (2.42). ∎

In the rest of this section, we develop the differential calculus on $\mathscr{P}_1(M)$ to which the preceding integration by parts formula applies.

Let $C^1(\mathbb{R};(\mathscr{O}(M)))$ denote the space of continuously differentiable maps $s\in\mathbb{R}\mapsto\mathfrak{P}(s)\in\mathscr{P}_1(\mathscr{O}(M))$ with property that

$$t\in[0,1]\mapsto[\omega(\mathfrak{P}'(0))](t)\equiv\omega([\mathfrak{P}'(0)](t))\in\mathbb{R}^d \text{ is an element of } \mathbf{H}_0$$

and

$$t\in[0,1]\mapsto[\phi(\mathfrak{P}'(0))](t)\equiv\phi([\mathfrak{P}'(0)](t))\in\mathfrak{o}(d)$$

$$\text{is continuous and vanishes at } t=0.$$

Second, given a separable Hilbert space X, let $\mathscr{M}(X)$ be the space of totally finite, X-valued Borel measures on $[0,1]$, and give $\mathscr{M}(X)$ the strong topology. We will write $F\in C^1(\mathscr{P}_1(\mathscr{O}(M));X)$ if $F\in C(\mathscr{P}_1(\mathscr{O}(M));X)$ and there exist

$$\mathfrak{D}F\in C(\mathscr{P}_1(\mathscr{O}(M));\mathbf{H}_0\otimes X) \qquad\text{and}\qquad \Lambda F\in C(\mathscr{P}_1(\mathscr{O}(M));\mathscr{M}(\mathfrak{o}(d)\otimes X))$$

with the property that, for all $\mathfrak{P}\in C^1(\mathscr{P}_1(\mathscr{O}(M)); X)$,

$$\frac{d}{ds}F(\mathfrak{P}(s))\bigg|_{s=0}=(\omega(\mathfrak{P}'(0)),\mathfrak{D}F(\mathfrak{P}(0)))_{\mathbf{H}_0}+\langle\phi(\mathfrak{P}'(0)),\Lambda F(\mathfrak{P}(0))\rangle,\quad(3.4)$$

where we have used $\langle\cdot,\cdot\rangle$ to denote the duality relation induced by integrating functions with respect to measures. It is an easy matter (cf. Lemma 4.15 in [ES]) to see that both $\mathfrak{D}F$ and ΛF are uniquely determined.

Because our real interest is in functions defined on $\mathscr{P}_1(M)$, we define $C^1(\mathscr{P}_1(M); X)$ to be the space of $f\in C(\mathscr{P}_1(M); X)$ with the property that $f\circ\pi\in C^1(\mathscr{P}_1(M)f; X)$. It is clear (cf. (4.19) in [ES]) that $\Lambda(f\circ\pi)\equiv\mathbf{0}$, and therefore

$$\frac{d}{ds}f\circ\pi(\mathfrak{P}(s))\bigg|_{s=0}=(Df(\mathfrak{P}(0)),\omega(\mathfrak{P}'(0)))_{\mathbf{H}_0}\qquad\text{where}\quad Df\equiv\mathfrak{D}(f\circ\pi).$$

Notice that, although Df is defined on $\mathscr{P}_1(\mathscr{O}(M))$ and not on $\mathscr{P}_1(M)$, for any pair $(f,g)\in C^1(\mathscr{P}_1(M); X)$, the inner product

$$(Df,Dg)_{\mathbf{H}_0}(p)\equiv(Df(\mathfrak{p}),Dg(\mathfrak{p}))_{\mathbf{H}_0},\qquad p=\pi\circ\mathfrak{p},\quad(3.5)$$

is well-defined. Next, let $\mathbf{D}^1(\mathscr{P}_1(M)f; X)$ denote the space of $f\in C^1(\mathscr{P}_1(M); X)$ for which there exists a continuous map $\lambda_f\in C(\mathscr{P}_1(\mathscr{O}(M)); \mathscr{M}(\mathbb{R}^d\otimes X))$ such that

$$[Df(\mathfrak{p})](t)=\int_{[0,1]}(t\wedge\tau-t\tau)\,\lambda_f(\mathfrak{p})(d\tau),\qquad t\in[0,1].$$

Clearly,

$$\pi\circ\mathfrak{q}=\pi\circ\mathfrak{p}\Rightarrow\lambda_f(\mathfrak{q})(d\tau)=\mathfrak{q}(\tau)^{-1}\,\mathfrak{p}(\tau)\,\lambda_f(\mathfrak{p})(d\tau).$$

In particular, if $\mathbf{D}^2(\mathscr{P}_1(M); X)$ stands for the space of functions $f\in\mathbf{D}^1(\mathscr{P}_1(M); X)$ for which $Df\in C^1(\mathscr{P}_1(M)f; \mathbf{H}_0\otimes X)$, then, for $\phi\in C([0,1]; \mathfrak{o}(d))$,

$$\langle\phi,\Lambda Df(\mathfrak{p})\rangle(t)=-\int_{[0,1]}(t\wedge\tau-t\tau)\,\phi(\tau)\,\lambda_f(\mathfrak{p})(d\tau).\quad(3.6)$$

Finally, let $C_0(\mathscr{P}_1(M); X)$ denote the set of bounded $f\in C(\mathscr{P}_1(M); X)$ which are $\mathscr{B}_T(M)$-measurable for some $T\in[0,1)$, and let $C_0(\mathscr{P}_1(\mathscr{O}(M)); \mathscr{M}_0(X))$ be the set of $\Lambda\in C(\mathscr{P}_1(\mathscr{O}(M)); \mathscr{M}(X))$ such that Λ has uniformly bounded total variation and there is a $T\in[0,1)$ for which Λ is $\mathscr{B}_T(M)$-measurable and $\Lambda(\mathfrak{p})((T,1])\equiv\mathbf{0}$. Then, we use $\mathbf{D}^1_0(\mathscr{P}_1(M);X)$ to denote the subset of $\mathbf{D}^1(\mathscr{P}_1(M);X)\cap C_0(\mathscr{P}_1(M);X)$ for which $\lambda_f\in C_0(\mathscr{O}(M);\mathfrak{o}(d)\times X)$; and we take $\mathbf{D}^2_0(\mathscr{P}_1(M);X)$ to be the subset of $f\in\mathbf{D}^1_0(\mathscr{P}_1(M);X)\cap\mathbf{D}^2(\mathscr{P}_1(M);X)$ for which $\mathfrak{D}\,Df$ is not only bounded but also, for any orthonormal basis $\{\mathbf{h}_\alpha\}$ in $\mathbf{H}_0$,

$$\sum_\alpha \sup_{\mathfrak{p}\in\mathscr{P}_1(M)f}\|(\mathbf{h}_\alpha\otimes\mathbf{h}_\alpha,\mathfrak{D}\,Df(\mathfrak{p}))_{\mathbf{H}_0\otimes\mathbf{H}_0}\|_X<\infty.$$

Clearly, when $f\in D^2_0(\mathscr{P}_1(M);X)$,

$$\begin{aligned}\mathfrak{p}\in\mathscr{P}_1(\mathscr{O}(M))\mapsto \operatorname{Trace}_{\mathbf{H}_0}&(\mathfrak{D}\,Df(\mathfrak{p}))\\ &\equiv\sum_\alpha(\mathbf{h}_\alpha\otimes\mathbf{h}_\alpha,\mathfrak{D}\,Df(p))_{\mathbf{H}_0\otimes\mathbf{H}_0}\in X\end{aligned}$$

is a bounded continuous function which does not depend on the choice of orthonormal basis $\{\mathbf{h}_\alpha\}$. Moreover,

$$\begin{aligned}\pi\circ\mathfrak{p}=\pi\circ\mathfrak{q}\qquad\text{and}\qquad \mathfrak{q}(t)^{-1}\,\mathfrak{p}(t)\qquad\text{independent of } t\\ \Rightarrow\operatorname{Trace}_{\mathbf{H}_0}(\mathfrak{D}\,Df(\mathfrak{q}))=\operatorname{Trace}_{\mathbf{H}_0}(\mathfrak{D}\,Df(\mathfrak{p})).\end{aligned}$$

3.7. Lemma. *There is a progressively measurable function* $\tilde{\mathfrak{R}}_{(\mathfrak{e},y)}:[0,1]\times\tilde{\mathfrak{W}}\to C([0,1];\mathbb{R}^d)$ *such that*

$$T\in[0,1]\mapsto\tilde{\mathfrak{R}}_{(\mathfrak{e},y)}(T,\tilde{\mathbf{w}})\in C([0,1];\mathbb{R}^d)\text{ is continuous}$$

and (cf. (2.22))

$$[\tilde{\mathfrak{R}}_{(\mathfrak{e},y)}(T,\tilde{\mathbf{w}})](t)=\int_0^T(t\wedge\tau-t\tau)\,\widetilde{\operatorname{Ric}}(\tilde{\mathfrak{F}}_{(\mathfrak{e},y)}(\tau,\tilde{\mathbf{w}}))\,d\mathbf{w}(\tau),\qquad t\in[0,1],\tag{3.8}$$

for $\tilde{\mu}$-almost every $\tilde{\mathbf{w}}\in\tilde{\mathfrak{W}}$. In fact, for each $p\in[2,\infty)$,

$$\mathbb{E}^{\tilde{\mu}}[\sup_{T\in[0,1]}\|\tilde{\mathfrak{R}}_{(\mathfrak{e},y)}(T)\|^p_{C([0,1];\mathbb{R}^d)}]<\infty,\tag{3.9}$$

and

$$T\in[0,1)\Rightarrow\mathbb{E}^{\tilde{\mu}}[\|\tilde{\mathfrak{R}}_{(\mathfrak{e},y)}(T)\|^p_{\mathbf{H}_0}]<\infty.\tag{3.10}$$

In particular,

$$(\mathbf{h},\tilde{\mathfrak{R}}_{(\mathfrak{e},y)}(T,\tilde{\mathbf{w}}))_{\mathbf{H}_0}=\int_0^T(\widetilde{\operatorname{Ric}}(\tilde{\mathfrak{F}}_{(\mathfrak{e},y)}(t,\tilde{\mathbf{w}}))\,\mathbf{h}(t),d\mathbf{w}(t))_{\mathbb{R}^d},\qquad T\in[0,1)\quad\text{and}\quad\mathbf{h}\in\mathbf{H}_0,\tag{3.11}$$

for $\tilde{\mu}$-almost every $\tilde{\mathbf{w}}\in\tilde{\mathfrak{W}}$.

Proof. After decomposing the right hand side of (3.8) into

$$\int_0^{T\wedge t} \tau(1-\tau)\,\widetilde{\mathrm{Ric}}(\mathfrak{F}_{(\mathrm{e},\,y)}(\tau,\tilde{\mathbf{w}}))\,d\mathbf{w}(\tau) - \int_0^T \tau(t-\tau)^+\,\widetilde{\mathrm{Ric}}(\mathfrak{F}_{(\mathrm{e},\,y)}(\tau,\tilde{\mathbf{w}}))\,d\mathbf{w}(\tau)$$

$$+\,t\int_{T\wedge t}^{T} (1-\tau)\,\widetilde{\mathrm{Ric}}(\mathfrak{F}_{(\mathrm{e},\,y)}(\tau,\tilde{\mathbf{w}}))\,d\mathbf{w}(\tau)$$

and using the estimate from (0.3) in [S], we see that everything except (3.11) follows from standard martingale theory plus an application of Kolmogorov's continuity criterion. To prove (3.11), simply note that, for any $\lambda\in\mathscr{M}(\mathbb{R}^d)$ and $\mathbf{h}\in\mathbf{H}_0$,

$$\langle\mathbf{h}(t),\lambda\rangle=(\mathbf{h},\mathbf{h}_\lambda)_{\mathbf{H}_0}\qquad\text{where}\quad \mathbf{h}_\lambda(t)\equiv\int_{[0,1]}(t\wedge\tau-t\tau)\,\lambda(d\tau).\quad\blacksquare\tag{3.12}$$

3.13. THEOREM. *Define* $\mathscr{U}_{(\mathrm{e},\,y)}\colon\tilde{\mathfrak{W}}\to C([0,1];\mathbb{R}^d)$ by

$$\begin{aligned}[\mathscr{U}_{(\mathrm{e},\,y)}(\tilde{\mathbf{w}})](t)&=\mathbf{w}(t)-t\mathbf{w}(1)-[\mathfrak{R}_{(\mathrm{e},\,y)}(\tilde{\mathbf{w}})](t)+[\mathfrak{R}_{(\mathrm{e},\,y)}(1,\tilde{\mathbf{w}})](t)\\&\quad+\int_0^t \tau(1-t)[\mathrm{Ric}\,\nabla(\log p_{1-\tau}(\cdot,y))-\tfrac14\nabla_k]\\&\quad\times(\mathfrak{F}_{(\mathrm{e},\,y)}(\tau,\tilde{\mathbf{w}}))\,d\tau,\end{aligned}\tag{3.14}$$

where $\kappa\colon M\to\mathbb{R}$ *given by*

$$\kappa(y)\equiv\mathrm{Trace}_{\mathbb{R}^d}(\mathrm{Ric}(\mathfrak{f})),\qquad \mathfrak{f}\in\pi^{-1}(\mathfrak{f}),$$

is the scalar curvature and

$$[\mathfrak{R}_{(\mathrm{e},\,y)}(\tilde{\mathbf{w}})](t)\equiv\int_0^t \tau(1-t)\,\mathrm{Ric}(\mathfrak{F}_{(\mathrm{e},\,y)}(\tau,\tilde{\mathbf{w}}))\,d\mathbf{w}(\tau).$$

Then, for any φ *and* f *from* $\mathbf{D}_0^1(\mathscr{P}_1(M);\mathbb{R})$ *and* $g\in\mathbf{D}_0^2(\mathscr{P}_1(M);\mathbb{R})$ (*cf.* (3.5)):

$$\begin{aligned}-\mathbb{E}^{P_{(x,y)}}[\varphi(Df,Dg)_{\mathbf{H}_0}]&=\mathbb{E}^{P_{(x,y)}}[f(D\varphi,Dg)_{\mathbf{H}_0}]\\&\quad+\mathbb{E}^{\tilde{\mu}}[((\varphi f)\circ\pi)(\mathfrak{F}_{(\mathrm{e},\,y)})\,\mathfrak{L}_{(\mathrm{e},\,y)}\,g(\tilde{\mathbf{w}})],\end{aligned}\tag{3.15}$$

where

$$\mathfrak{L}_{(\mathrm{e},\,y)}\,g(\tilde{\mathbf{w}})\equiv\mathrm{Trace}_{\mathbf{H}_0}(\mathfrak{D}\,Dg(\mathfrak{F}_{(\mathrm{e},\,y)}(\cdot,\tilde{\mathbf{w}})))-\langle\mathscr{U}_{(\mathrm{e},\,y)},\lambda_g(\mathfrak{F}_{(\mathrm{e},\,y)}(\cdot,\tilde{\mathbf{w}}))\rangle.\tag{3.16}$$

Proof. Starting from (3.3), the proof of this result is very much like that of Theorem 4.30 in [ES], and so we will stress only the places where it differs.

For $n \geqslant 1$, set

$$h_n(t) = \frac{\sqrt{2}\sin(n\pi t)}{n\pi}, \qquad t \in [0, 1].$$

Then, because $\{h_n\}_1^\infty$ forms an orthonormal basis in $\mathbf{H}_0$ (when $d = 1$), for any $\lambda \in \mathscr{M}(\mathbb{R})$ (cf. (3.12))

$$h_\lambda = \sum_{n=1}^{\infty} (h_\lambda, h_n)_{\mathbf{H}_0} = \sum_1^\infty \langle h_n, \lambda \rangle h_n,$$

where the convergence is in $\mathbf{H}_0$. Hence, it is an easy matter to check that

$$\sum_1^\infty h_n(t)\, h_n(\tau) = t \wedge \tau - t\tau, \qquad (t, \tau) \in [0, 1]^2, \tag{3.17}$$

where the convergence is absolute and uniform on $[0, 1]^2$. Next, take $\mathbf{h}_\alpha = h_n \mathbf{e}_k$ for $\alpha = (n, k)$, $1 \leqslant k \leqslant d$. Then $\{\mathbf{h}_\alpha : \alpha \in \mathbb{Z}^+ \times \{1, ..., d\}\}$ forms an orthonormal basis in $\mathbf{H}_0$ and

$$\sum_\alpha \mathbf{h}_\alpha \otimes \mathbf{h}_\alpha = (t \wedge \tau - t\tau)\mathbf{I},$$

where, again, the convergence is absolute and uniform.

Choose $T \in [0, 1)$ so that φ, f, and g are all $\mathscr{B}_T$-measurable and $\lambda_g(\mathfrak{p})$ is supported on $[0, T]$ for all $\mathfrak{p} \in \mathscr{P}_1(\mathscr{O}(M))$. For each α, set

$$F_\alpha(\mathfrak{p}) = (Df(\mathfrak{p}), \mathbf{h}_\alpha)_{\mathbf{H}_0} \qquad \text{and} \qquad G_\alpha(\mathfrak{p}) = (Dg(\mathfrak{p}), \mathbf{h}_\alpha)_{\mathbf{H}_0}, \qquad \mathfrak{p} \in \mathscr{P}_1(\mathscr{O}(M)).$$

Then the left hand side of (3.15) is equal to the sum over α of

$$-\mathbb{E}^{\tilde{\mu}}[(\varphi \circ \pi)(\mathfrak{F}_{(\mathrm{e},\, \mathrm{y})})(F_\alpha G_\alpha)(\mathfrak{F}_{(\mathrm{e},\, \mathrm{y})})],$$

which, by (3.3), is equal to

$$\begin{aligned}
&\mathbb{E}^{\tilde{\mu}}[(f \circ \pi)(\mathfrak{F}_{(\mathrm{e},\, y)})(D\varphi(\mathfrak{F}_{(\mathrm{e},\, y)}), \mathbf{h}_\alpha)_{\mathbf{H}_0}\,(Df(\mathfrak{F}_{(\mathrm{e},\, y)}), \mathbf{h}_\alpha)_{\mathbf{H}_0}] \\
&\quad + \mathbb{E}^{\tilde{\mu}}[(\varphi f) \circ \pi(\mathfrak{F}_{(\mathrm{e},\, y)})(\mathfrak{D}\, Dg(\mathfrak{F}_{(\mathrm{e},\, y)}), \mathbf{h}_\alpha \otimes \mathbf{h}_\alpha)_{\mathbf{H}_0}] \\
&\quad + \mathbb{E}^{\tilde{\mu}}[(\varphi f) \circ \pi(\mathfrak{F}_{(\mathrm{e},\, y)})\langle \phi_\alpha \mathbf{h}_\alpha, \lambda_g(\mathfrak{F}_{(\mathrm{e},\, y)})\rangle] \\
&\quad - \mathbb{E}^{\tilde{\mu}}\left[((\varphi f) \circ \pi G_\alpha)(\mathfrak{F}_{(\mathrm{e},\, y)}) \int_0^1 (\dot{\mathbf{h}}_\alpha(t) + \widetilde{\mathrm{Ric}}(t, \mathfrak{F}_{(\mathrm{e},\, y)}(t))\, \mathbf{h}_\alpha(t), d\mathbf{w}(t))_{\mathbb{R}^d}\right],
\end{aligned}$$

where (cf. (2.12))

$$[\phi_\alpha(\tilde{\mathbf{w}})](t) = \sum_{l, l'=1}^{d} \int_0^t \mathbf{h}_\alpha(\tau)\, \Phi_{l,l'}(\mathfrak{F}_{(\mathrm{e},\, y)}(\tau, \mathbf{w})) \circ d_\tau[\Theta_{(\mathrm{e},\, y, \mathbf{h}_\alpha)}(\tau, \tilde{\mathbf{w}})](0)_l.$$

After summing the first and second terms with respect to α, one gets, respectively,

$$\mathbb{E}^{\tilde{\mu}}[f \circ \pi(\mathfrak{F}_{(\mathbf{e}, y)})(D\varphi(\mathfrak{F}_{(\mathbf{e}, y)}), Dg(\mathfrak{F}_{(\mathbf{e}, y)}))_{\mathbf{H}_0}]$$

and

$$\mathbb{E}^{\tilde{\mu}}[(\varphi f) \circ \pi(\mathfrak{F}_{(\mathbf{e}, y)}) \operatorname{Trace}_{\mathbf{H}_0}(\mathfrak{D}\, Dg(\mathfrak{F}_{(\mathbf{e}, y)}))].$$

To handle the third term, first note that, for each n,

$$\sum_{k=1}^{d} [\phi_{(n,k)}(\tilde{\mathbf{w}})](t)\, \mathbf{h}_{(n,k)}(t)$$

$$= h_n(t) \int_0^t h_n(\tau) \operatorname{Ric}(\mathfrak{F}_{(\mathbf{e}, y)}(\tau, \tilde{\mathbf{w}}))\, d\mathbf{w}(\tau)$$

$$+ h_n(t) \int_0^t h_n(\tau)[\tfrac{1}{4} \nabla\kappa - \operatorname{Ric} \nabla(\log p_{1-\tau}(\cdot, y))](\mathfrak{F}_{(\mathbf{e}, y)}(\tau, \tilde{\mathbf{w}}))\, d\tau,$$

where we have used (4.29) in [ES] to convert the Stratonovich integral into an Itô one. Hence, because $\lambda_g(\cdot)$ is supported on $[0, T]$, one can use (3.17) to see that the sum of the third term with resect to α gives

$$\mathbb{E}^{\tilde{\mu}}\Bigg[(\varphi f) \circ \pi(\mathfrak{F}_{(\mathbf{e}, y)}) \Bigg(\mathfrak{R}_{(\mathbf{e}, y)}(t) + \int_0^t \tau(1-t)$$

$$\times [\tfrac{1}{4} \nabla\kappa - \nabla \log p_{1-\tau}(\cdot, y)](\mathfrak{F}_{(\mathbf{e}, y)}(\tau))\, \lambda_g(\mathfrak{F}_{(\mathbf{e}, y)})(dt)\Bigg)\Bigg].$$

Turning to the fourth term, note that

$$G_\alpha(\mathfrak{F}_{(\mathbf{e}, y)}(\cdot, \tilde{\mathbf{w}})) \int_0^1 (\dot{\mathbf{h}}_\alpha(t), d\mathbf{w}(t))_{\mathbb{R}^d} = X_\alpha(\mathbf{w}) \langle \mathbf{h}_\alpha, \lambda_g(\mathfrak{F}_{(\mathbf{e}, y)}(\cdot, \tilde{\mathbf{w}}))\rangle,$$

where $X_\alpha(\mathbf{w}) \equiv \int_0^1 (\mathbf{h}_\alpha(t), d\mathbf{w}(t))_{\mathbb{R}^d}$. Hence, since $\sum_\alpha X_\alpha(\mathbf{w})\, \mathbf{h}_\alpha(t)$ converges uniformly with respect to $t \in [0, 1]$ to $\mathbf{w}(t) - t\mathbf{w}(1)$ in $L^p(\mu)$ for every $p \in [1, \infty)$, it follows that

$$\sum_\alpha \mathbb{E}^{\tilde{\mu}}\left[((\varphi f) \circ \pi G_\alpha)(\mathfrak{F}_{(\mathbf{e}, y)}) \int_0^1 (\dot{\mathbf{h}}_\alpha(t), d\mathbf{w}(t))_{\mathbb{R}^d}\right]$$

$$= \mathbb{E}^{\tilde{\mu}}\left[(\varphi f) \circ \pi(\mathfrak{F}_{(\mathbf{e}, y)}) \int_0^1 (\mathbf{w}(t) - t\mathbf{w}(1), \lambda_g(\mathfrak{F}_{(\mathbf{e}, y)})(dt))_{\mathbb{R}^d}\right].$$

Finally, to handle the remaining part of the fourth term, note that, by (3.11),

$$\mathbb{E}^{\tilde{\mu}}\left[((\varphi f)\circ\pi G_\alpha)(\mathfrak{F}_{(e,y)})\int_0^1 \widetilde{\mathrm{Ric}}(\mathfrak{F}_{(e,y)}(t))\,\mathbf{h}_\alpha(t)\,d\mathbf{w}(t)\right]$$

$$=\mathbb{E}^{\tilde{\mu}}\left[((\varphi f)\circ\pi G_\alpha)(\mathfrak{F}_{(e,y)})\int_0^T \widetilde{\mathrm{Ric}}(\mathfrak{F}_{(e,y)}(t))\,\mathbf{h}_\alpha(t)\,d\mathbf{w}(t)\right]$$

$$=\mathbb{E}^{\tilde{\mu}}[((\varphi f)\circ\pi)(\mathfrak{F}_{(e,y)})(Dg(\mathfrak{F}_{(e,y)}),\mathbf{h}_\alpha)_{\mathbf{H}_0}(\tilde{\mathfrak{R}}_{(e,y)}(T),\mathbf{h}_\alpha)_{\mathbf{H}_0}]$$

for each α. Hence, after summing with respect to α and again applying (3.11), we arrive first at

$$\mathbb{E}^{\tilde{\mu}}[(\varphi f)\circ\pi(\mathfrak{F}_{(e,y)})\langle\tilde{\mathfrak{R}}_{(e,y)}(T),\lambda_g(\mathfrak{F}_{(e,y)}\rangle]$$

and then at

$$\mathbb{E}^{\tilde{\mu}}[(\varphi f)\circ\pi(\mathfrak{F}_{(e,y)})\langle\tilde{\mathfrak{R}}_{(e,y)}(1),\lambda_g(\mathfrak{F}_{(e,y)}\rangle]. \quad \blacksquare$$

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